Quantum Programs as Kleisli Maps

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https://bram.westerbaan.name/kleisli.pdf

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Substochastic map:

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Kleisli Maps

$$\frac{X \xrightarrow{\text{substochastic}} Y}{X \xrightarrow{\text{("sharp") map}} \mathcal{S}(Y)}$$

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So substochastic maps are "Kleisli maps".

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YES! when A, B, and $\mathcal{Q}(B)$ may have **infinite** dimension.

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4. $\mathbf{vN} \subseteq \mathbf{vN}_{\mathrm{CPsU}}$ is the (wide) subcategory of multiplicative unital maps — the “sharp” quantum processes.

Main Theorem

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By the Adjoint Functor Theorem it suffices to show that

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Point 2 follows from this: if $\mathscr{B}$ is a von Neumann subalgebra generated by a subset X of a von Neumann algebra, then

$$\#\mathscr{B} \;\leq\; 2^{2^{\#\mathbb{C}\,\cdot\,\#X}}. \qquad \square$$

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   At least $\mathcal{F}(\mathbb{C}^2)$ is not commutative (because $f(1,0)$ and $f(0,1)$ might not commute for a CPsU-map $f\colon \mathbb{C}^2 \to \mathscr{A}$).

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