In search of the spectrum

Tom Leinster

University of Edinburgh

Meanings of ‘spectrum’

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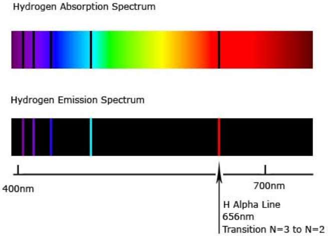
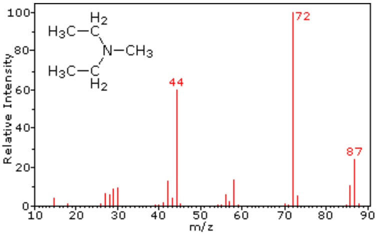
Physics & chemistry: emission/absorption spectra, mass spectroscopy, etc.





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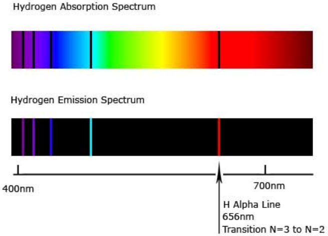

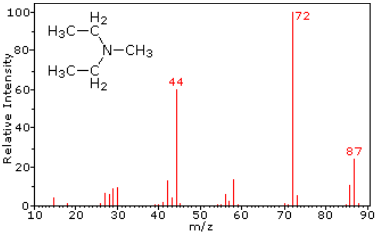

Linear algebra: eigenvalues of an operator, with their algebraic multiplicities

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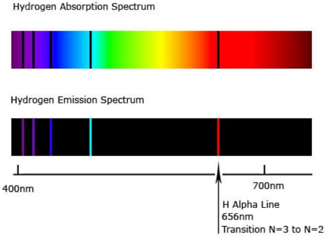

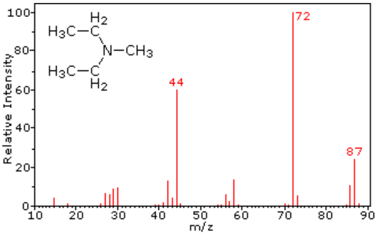

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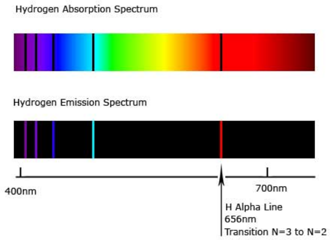

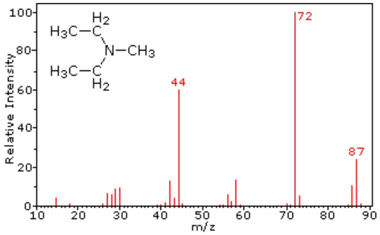

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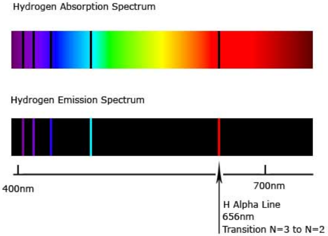

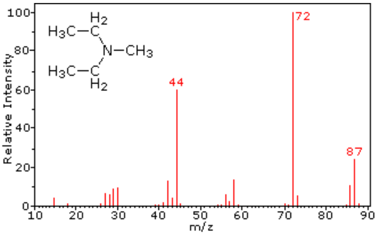

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. . . and many more meanings, all related to one another.

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This talk offers one.

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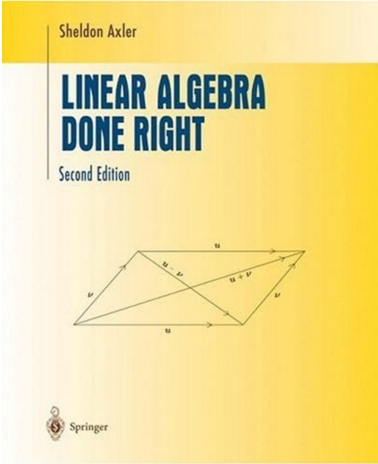
1. Linear Algebra Done Right

Linear Algebra Done Right
by Sheldon Axler

Sheldon Axler (1975)



Book (1996)

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Contrast: usually $X \neq \ker(T) + \operatorname{im}(T)$, although we do have $\dim X = \dim\ker(T) + \dim\operatorname{im}(T)$.

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The eventual eigenspace $\ker^\infty(T - \lambda)$ is trivial for all except finitely many values of $\lambda \in k$ — namely, the eigenvalues.

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All have their usual meanings!

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- maps $X \circlearrowleft T \longrightarrow Y \circlearrowleft S$ are maps $f: X \longrightarrow Y$ in $\mathscr{C}$ such that $S \circ f = f \circ T$.

Let **FDVect** be the category of finite-dimensional vector spaces.

We're interested in Endo(**FDVect**), the category of linear operators.

There is a functor

$$\begin{array}{ccc} \mathrm{Endo}(\mathbf{FDVect}) & \longrightarrow & \mathrm{Endo}(\mathbf{FDVect}) \\ X \circlearrowleft T & \mapsto & \mathrm{im}^\infty(T) \circlearrowleft T^\times. \end{array}$$

On maps, it's defined by restriction: any map of operators $f: X \circlearrowleft T \longrightarrow Y \circlearrowleft S$ restricts to a map $f: \mathrm{im}^\infty(T) \circlearrowleft T^\times \longrightarrow \mathrm{im}^\infty(S) \circlearrowleft S^\times$.

2. Invariants

Definition and examples

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Let $\mathscr{E}$ be a category. An invariant of objects of $\mathscr{E}$ is a function

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  (Can describe these iso types concretely via Jordan normal form.)

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- because of cyclicity...

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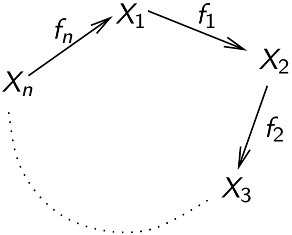
A cyclic invariant Φ assigns a value to any cycle



in $\mathscr{C}$, since $\Phi(f_i \circ \cdots \circ f_1 \circ f_n \circ \cdots f_{i+1})$ is independent of i.

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(In fact, this is the *initial* cyclic invariant of linear operators.)

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So, the multiplicity of 0 as an eigenvalue is not a cyclic invariant.

4. Balanced invariants

Looking at an object from all directions

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In other words, Spec and $\text{Spec}^\times$ are balanced invariants of linear operators.

5. The theorem

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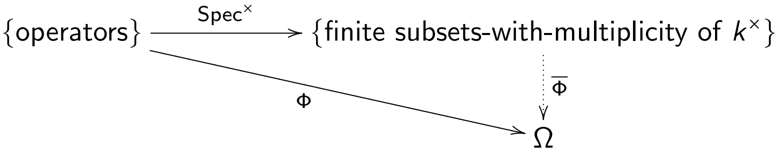

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$$\begin{array}{ccc} \{\text{operators}\} & \xrightarrow{\ \mathrm{Spec}^\times\ } & \{\text{finite subsets-with-multiplicity of } k^\times\} \\ & \searrow^{\Phi} & \Big\downarrow{\scriptstyle \overline{\Phi}} \\ & & \Omega \end{array}$$

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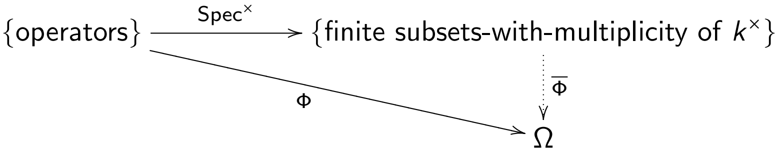

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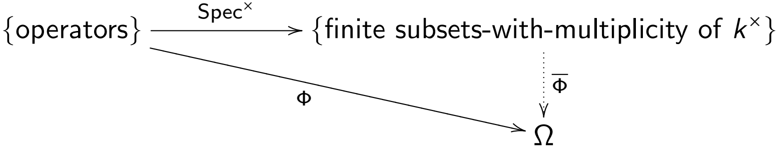

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E.g.: tr is cyclic and balanced, and indeed $\mathrm{tr}(T) = \sum_{\lambda \in k^\times} \alpha_T(\lambda) \cdot \lambda$.

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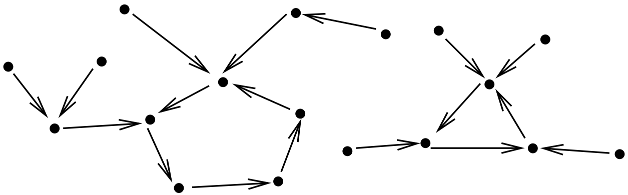

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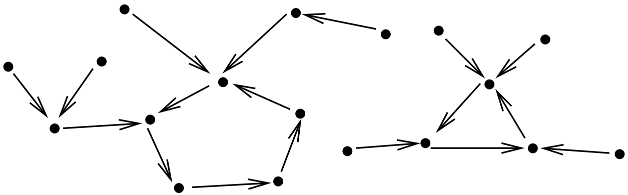

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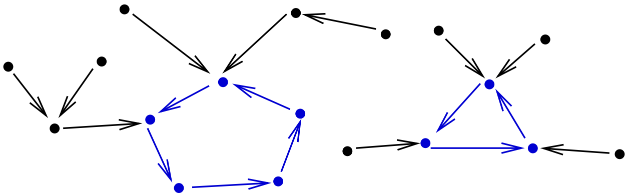

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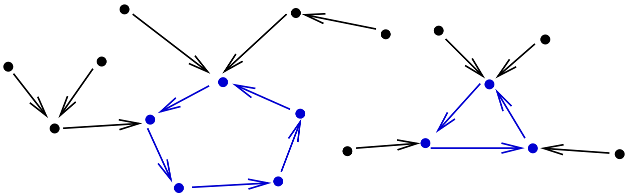

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That's the 'invertible spectrum' of an operator on a finite set.

Postscript: Commutative rings and topos theory

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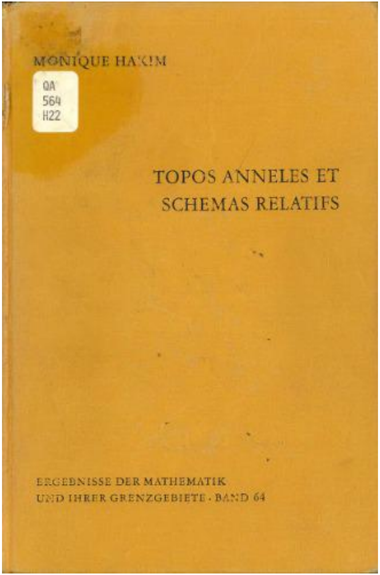
But how can we understand the ring-theoretic spectrum abstractly?

Hakim's theorem

Monique Hakim (1986)



Book (1972)

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You can think of the adjoint as constructing the 'free local ring' on a ring: *but* it might live in a different topos from the ring you started with.

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In this sense, Spec *is exactly that right adjoint*.

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- The characterization of the spectrum of an operator coming from Hakim's theorem is less direct than the one established in this talk, which stays within the topos of sets.