Von Neumann Algebras Form a Model for the Quantum Lambda Calculus

arXiv:1603.02133 [cs.LO]

Kenta Cho   Abraham Westerbaan

Radboud University, Nijmegen

iCIS | Digital Security
Radboud University

QPL 2016
9 June 2016

We present ...

A denotational model for

the Quantum Lambda Calculus
[Selinger & Valiron 2000s]

by

von Neumann Algebras
[von Neumann (with Murray) ’30s–’40s]

We present ...

A denotational model for

higher-order quantum programming language

the Quantum Lambda Calculus

[Selinger & Valiron 2000s]

by

von Neumann Algebras

[von Neumann (with Murray) '30s–'40s]

We present ...

A denotational model for

higher-order quantum programming language

the Quantum Lambda Calculus

[Selinger & Valiron 2000s]

by

generalisation of matrix algebras e.g. $\mathcal{M}_n = \mathbb{C}^{n \times n}$

von Neumann Algebras

[von Neumann (with Murray) '30s–'40s]

Quantum Lambda Calculus

Quantum Lambda Calculus
$\approx$ linear lambda calculus + quantum primitives

Quantum Lambda Calculus

Quantum Lambda Calculus
$\approx$ linear lambda calculus + quantum primitives

qubit type, preparation, measurement, unitary transformation

Quantum Lambda Calculus

Quantum Lambda Calculus
$\approx$ linear lambda calculus + quantum primitives

qubit type, preparation, measurement, unitary transformation

(Unlike **Quipper**, no manipulation of quantum circuits)

Quantum Lambda Calculus

Quantum Lambda Calculus
$\approx$ linear lambda calculus + quantum primitives

qubit type, preparation, measurement, unitary transformation

(Unlike **Quipper**, no manipulation of quantum circuits)

- Type system is based on linear logic with the exponential modality "!"
    - Each input can be used only (at most) once, unless it has a duplicable type $!A$

Quantum Lambda Calculus

Quantum Lambda Calculus
≈ linear lambda calculus + quantum primitives

qubit type, preparation, measurement, unitary transformation

(Unlike **Quipper**, no manipulation of quantum circuits)

- Type system is based on linear logic with the exponential modality “!”
  - Each input can be used only (at most) once, unless it has a duplicable type $!A$
- Studied extensively by Selinger and Valiron in 2000s

Syntax of Quantum Lambda Calculus

We follow [Selinger & Valiron '06, '09] (with $\oplus$ type, without recursion)

$$
\begin{array}{ll}
\textit{Type} & A, B ::= \top \mid \texttt{qbit} \mid\, !A \mid A \multimap B \mid A \otimes B \mid A \oplus B \\
\textit{Term} & M, N, L ::= x \mid * \mid \texttt{new} \mid \texttt{meas} \mid U \mid \lambda x.M \mid M N \\
& \qquad\qquad \mid \texttt{let}\ \langle x, y \rangle = N\ \texttt{in}\ M \\
& \qquad\qquad \mid \langle M, N \rangle \mid \texttt{inl}(M) \mid \texttt{inr}(N) \\
& \qquad\qquad \mid \texttt{match}\ L\ \texttt{with}\ (x \mapsto M \mid y \mapsto N)
\end{array}
$$

Syntax of Quantum Lambda Calculus

We follow [Selinger & Valiron '06, '09] (with $\oplus$ type, without recursion)

$$\textit{Type}\quad A, B ::= \top \mid \texttt{qbit} \mid !A \mid A \multimap B \mid A \otimes B \mid A \oplus B$$

$$\begin{aligned}\textit{Term}\quad M, N, L ::= {} & x \mid * \mid \texttt{new} \mid \texttt{meas} \mid U \mid \lambda x.M \mid M N \\ & \mid \texttt{let}\ \langle x, y\rangle = N\ \texttt{in}\ M \\ & \mid \langle M, N\rangle \mid \texttt{inl}(M) \mid \texttt{inr}(N) \\ & \mid \texttt{match}\ L\ \texttt{with}\ (x \mapsto M \mid y \mapsto N)\end{aligned}$$

Syntax of Quantum Lambda Calculus

We follow [Selinger & Valiron '06, '09] (with $\oplus$ type, without recursion)

$$\begin{array}{lll} \textit{Type} & A, B ::= & \top \mid \texttt{qbit} \mid {!A} \mid A \multimap B \mid A \otimes B \mid A \oplus B \\ \textit{Term} & M, N, L ::= & x \mid * \mid \texttt{new} \mid \texttt{meas} \mid U \mid \lambda x.M \mid MN \\ & & \mid \texttt{let}\ \langle x, y \rangle = N\ \texttt{in}\ M \\ & & \mid \langle M, N \rangle \mid \texttt{inl}(M) \mid \texttt{inr}(N) \\ & & \mid \texttt{match}\ L\ \texttt{with}\ (x \mapsto M \mid y \mapsto N) \end{array}$$

Examples of typing:

✓ $x : \texttt{qbit}, y : \texttt{qbit} \vdash \langle x, y \rangle : \texttt{qbit} \otimes \texttt{qbit}$

Syntax of Quantum Lambda Calculus

We follow [Selinger & Valiron '06, '09] (with $\oplus$ type, without recursion)

$$\textit{Type} \quad A, B ::= \top \mid \mathtt{qbit} \mid {!A} \mid A \multimap B \mid A \otimes B \mid A \oplus B$$

$$\begin{aligned}\textit{Term} \quad M, N, L ::= {} & x \mid * \mid \mathtt{new} \mid \mathtt{meas} \mid U \mid \lambda x.M \mid MN \\ & \mid \mathtt{let}\ \langle x, y\rangle = N\ \mathtt{in}\ M \\ & \mid \langle M, N\rangle \mid \mathtt{inl}(M) \mid \mathtt{inr}(N) \\ & \mid \mathtt{match}\ L\ \mathtt{with}\ (x \mapsto M \mid y \mapsto N)\end{aligned}$$

Examples of typing:

✓ $x : \mathtt{qbit}, y : \mathtt{qbit} \vdash \langle x, y\rangle : \mathtt{qbit} \otimes \mathtt{qbit}$

✗ $x : \mathtt{qbit} \vdash \langle x, x\rangle : \mathtt{qbit} \otimes \mathtt{qbit}$

Syntax of Quantum Lambda Calculus

We follow [Selinger & Valiron '06, '09] (with $\oplus$ type, without recursion)

Type $A, B ::= \top \mid \texttt{qbit} \mid\, !A \mid A \multimap B \mid A \otimes B \mid A \oplus B$

Term $M, N, L ::= x \mid * \mid \texttt{new} \mid \texttt{meas} \mid U \mid \lambda x.M \mid MN$

$\mid \texttt{let}\ \langle x, y \rangle = N\ \texttt{in}\ M$

$\mid \langle M, N \rangle \mid \texttt{inl}(M) \mid \texttt{inr}(N)$

$\mid \texttt{match}\ L\ \texttt{with}\ (x \mapsto M \mid y \mapsto N)$

Examples of typing:

✓ $x : \texttt{qbit}, y : \texttt{qbit} \vdash \langle x, y \rangle : \texttt{qbit} \otimes \texttt{qbit}$

✗ $x : \texttt{qbit} \vdash \langle x, x \rangle : \texttt{qbit} \otimes \texttt{qbit}$

✓ $x : !A \vdash \langle x, x \rangle : !A \otimes !A$

Models of Quantum Lambda Calculus

A (denotational/categorical) *model* of a language consists of a category $\mathbf{C}$ and an interpretation $[\![-]\!]$:

$$\begin{array}{ccc} \text{types } A & \longmapsto & \text{objects } [\![A]\!] \in \mathbf{C} \\ \begin{array}{c}\text{well-typed terms}\\ x : A \vdash M : B\end{array} & \longmapsto & \text{arrows } [\![A]\!] \xrightarrow{[\![M]\!]} [\![B]\!] \text{ in } \mathbf{C} \end{array}$$

Models of Quantum Lambda Calculus

A (denotational/categorical) *model* of a language consists of a category $\mathbf{C}$ and an interpretation $[\![-]\!]$:

$$\begin{array}{ccc} \text{types } A & \longmapsto & \text{objects } [\![A]\!] \in \mathbf{C} \\ \begin{array}{c}\text{well-typed terms} \\ x : A \vdash M : B\end{array} & \longmapsto & \text{arrows } [\![A]\!] \xrightarrow{[\![M]\!]} [\![B]\!] \text{ in } \mathbf{C} \end{array}$$

Models of quantum lambda calculi are nontrivial!

Models of Quantum Lambda Calculus

A (denotational/categorical) *model* of a language consists of a category $\mathbf{C}$ and an interpretation $[\![-]\!]$:

types A	$\longmapsto$	objects $[\![A]\!] \in \mathbf{C}$
well-typed terms $x : A \vdash M : B$	$\longmapsto$	arrows $[\![A]\!] \xrightarrow{[\![M]\!]} [\![B]\!]$ in $\mathbf{C}$

Models of quantum lambda calculi are nontrivial!

- Selinger & Valiron introduced a quantum lambda calculus with its operational semantics in 2005

Models of Quantum Lambda Calculus

A (denotational/categorical) *model* of a language consists of a category $\mathbf{C}$ and an interpretation $[\![-]\!]$:

types A	$\longmapsto$	objects $[\![A]\!] \in \mathbf{C}$
well-typed terms $x : A \vdash M : B$	$\longmapsto$	arrows $[\![A]\!] \xrightarrow{[\![M]\!]} [\![B]\!]$ in $\mathbf{C}$

Models of quantum lambda calculi are nontrivial!

- Selinger & Valiron introduced a quantum lambda calculus with its operational semantics in 2005
- The first model was obtained by Malherbe in 2010 using presheaf categories

Models of Quantum Lambda Calculus

A (denotational/categorical) *model* of a language consists of a category $\mathbf{C}$ and an interpretation $[\![-]\!]$:

types A $\longmapsto$ objects $[\![A]\!] \in \mathbf{C}$

well-typed terms $x : A \vdash M : B$ $\longmapsto$ arrows $[\![A]\!] \xrightarrow{[\![M]\!]} [\![B]\!]$ in $\mathbf{C}$

Models of quantum lambda calculi are nontrivial!

- Selinger & Valiron introduced a quantum lambda calculus with its operational semantics in 2005
- The first model was obtained by Malherbe in 2010 using presheaf categories
- Two other models (both accommodate recursion)
    - [Hasuo & Hoshino, LICS'11], via GoI
    - [Pagani, Selinger & Valiron, POPL'14], applying quantitative semantics

Previous and our approaches

> One reason that designing such a semantics [of QLC] is difficult is that quantum computation is inherently defined on finite dimensional Hilbert spaces, whereas the semantics of higher-order functional programming languages [...] is inherently infinitary. [Pagani, Selinger & Valiron '14]

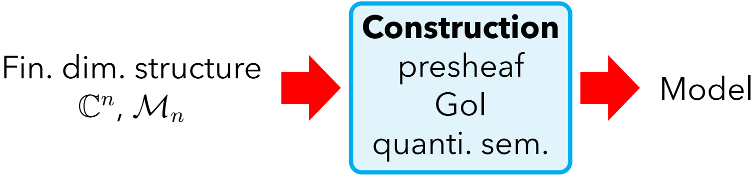
Previous and our approaches

> One reason that designing such a semantics [of QLC] is difficult is that quantum computation is inherently defined on finite dimensional Hilbert spaces, whereas the semantics of higher-order functional programming languages [...] is inherently infinitary. [Pagani, Selinger & Valiron '14]

Previous and our approaches

> One reason that designing such a semantics [of QLC] is difficult is that quantum computation is inherently defined on finite dimensional Hilbert spaces, whereas the semantics of higher-order functional programming languages [...] is inherently infinitary. [Pagani, Selinger & Valiron '14]

Previous approaches:



Fin. dim. structure
$\mathbb{C}^n$, $\mathcal{M}_n$

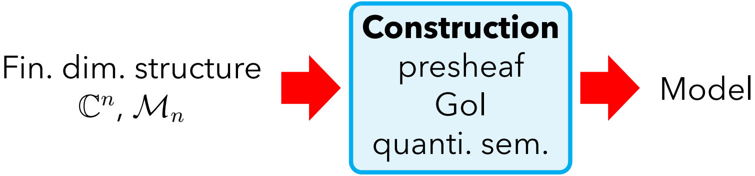
Construction
presheaf
GoI
quanti. sem.

Model

Previous and our approaches

> One reason that designing such a semantics [of QLC] is difficult is that quantum computation is inherently defined on finite dimensional Hilbert spaces, whereas the semantics of higher-order functional programming languages [...] is inherently infinitary. [Pagani, Selinger & Valiron '14]

Previous approaches:



Fin. dim. structure
$\mathbb{C}^n$, $\mathcal{M}_n$

Construction
presheaf
GoI
quanti. sem.

Model

Our approach: simply use von Neumann algebras, an infinite dimensional generalisation of matrix algebras

Von Neumann algebras

- A *von Neumann algebra* (aka. *W^*-algebra*) is a $*$-algebra (‘ring’) of operators on a Hilbert space which is closed in a suitable topology
- Developed by von Neumann and Murray in a series of papers “On rings of operators” in 1930s–1940s

Von Neumann algebras

- A *von Neumann algebra* (aka. W^**-algebra*) is a $*$-algebra ('ring') of operators on a Hilbert space which is closed in a suitable topology
- Developed by von Neumann and Murray in a series of papers "On rings of operators" in 1930s–1940s
- **Examples**: $\mathcal{B}(\mathcal{H})$; $\mathcal{M}_{n_1} \oplus \cdots \oplus \mathcal{M}_{n_k}$; $\mathbb{C}^n$

Von Neumann algebras

- A *von Neumann algebra* (aka. *W^*-algebra*) is a $*$-algebra ('ring') of operators on a Hilbert space which is closed in a suitable topology
- Developed by von Neumann and Murray in a series of papers "On rings of operators" in 1930s–1940s
- **Examples**: $\mathcal{B}(\mathcal{H})$; $\mathcal{M}_{n_1} \oplus \cdots \oplus \mathcal{M}_{n_k}$; $\mathbb{C}^n$

> [The theory of von Neumann algebras] generalizes many familiar facts about finite-dimensional algebra, and is currently one of the most powerful tools in the study of quantum physics. [P. R. Halmos 1973]

Interpretation of types in v.N. algebras

Type $A, B ::= \top \mid \mathtt{qbit} \mid\, !A \mid A \multimap B \mid A \otimes B \mid A \oplus B$

Interpretation of types in v.N. algebras

Type $A, B ::= \top \mid \texttt{qbit} \mid !A \mid A \multimap B \mid A \otimes B \mid A \oplus B$

$[\![\top]\!] = \mathbb{C}$	complex numbers
$[\![\texttt{qbit}]\!] = \mathcal{M}_2$	2×2 matrices
$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$	tensor product of v.N. alg.
$[\![A \oplus B]\!] = [\![A]\!] \oplus [\![B]\!]$	direct sum of v.N. alg.

Interpretation of types in v.N. algebras

Type $A, B ::= \top \mid \texttt{qbit} \mid\, !A \mid A \multimap B \mid A \otimes B \mid A \oplus B$

$[\![\top]\!] = \mathbb{C}$	complex numbers
$[\![\texttt{qbit}]\!] = \mathcal{M}_2$	2×2 matrices
$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$	tensor product of v.N. alg.
$[\![A \oplus B]\!] = [\![A]\!] \oplus [\![B]\!]$	direct sum of v.N. alg.
$[\![A \multimap B]\!] = ??$	
$[\![!A]\!] = ??$	

Interpretation of types in v.N. algebras

Type $A, B ::= \top \mid \texttt{qbit} \mid !A \mid A \multimap B \mid A \otimes B \mid A \oplus B$

$[\![\top]\!] = \mathbb{C}$	complex numbers
$[\![\texttt{qbit}]\!] = \mathcal{M}_2$	2×2 matrices
$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$	tensor product of v.N. alg.
$[\![A \oplus B]\!] = [\![A]\!] \oplus [\![B]\!]$	direct sum of v.N. alg.
$[\![A \multimap B]\!] = (\mathcal{F}\mathcal{J}[\![B]\!])^{*[\![A]\!]}$	
$[\![!A]\!] = \ell^\infty(\mathbf{vN}([\![A]\!], \mathbb{C}))$	

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-,\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

Interpretation of types in v.N. algebras

Type $A, B ::= \top \mid \texttt{qbit} \mid !A \mid A \multimap B \mid A \otimes B \mid A \oplus B$

$[\![\top]\!] = \mathbb{C}$	complex numbers
$[\![\texttt{qbit}]\!] = \mathcal{M}_2$	2×2 matrices
$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$	tensor product of v.N. alg.
$[\![A \oplus B]\!] = [\![A]\!] \oplus [\![B]\!]$	direct sum of v.N. alg.
$[\![A \multimap B]\!] = (\mathcal{F}\mathcal{J}[\![B]\!])^{*[\![A]\!]}$	
$[\![!A]\!] = \ell^\infty(\mathbf{vN}([\![A]\!], \mathbb{C}))$	

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

How does this work?

Categorical structures for the QLC

A *concrete model of the QLC* [Selinger & Valiron '09] is:

$$\begin{tikzcd}
(\mathbf{C}, \otimes, I) \arrow[loop left, "L"] \arrow[loop right, "T"]
\end{tikzcd}$$

Categorical structures for the QLC

A *concrete model of the QLC* [Selinger & Valiron '09] is:

$$\begin{tikzcd} (\mathbf{C}, \otimes, I) \arrow[loop left, "L"] \arrow[loop right, "T"] \end{tikzcd}$$

- $(\mathbf{C}, \otimes, I)$ a SMC with finite coproducts $(\oplus, 0)$
- T a monad
- L a comonad

Categorical structures for the QLC

A *concrete model of the QLC* [Selinger & Valiron '09] is:

$$\begin{tikzcd}
(\mathbf{C}, \otimes, I) \arrow[loop left, "L"] \arrow[loop right, "T"]
\end{tikzcd}$$

- $(\mathbf{C}, \otimes, I)$ a SMC with finite coproducts $(\oplus, 0)$
- T a strong monad with a Kleisli exponential $\multimap$ s.t. $\mathbf{C}(A \otimes B, TC) \cong \mathbf{C}(A, B \multimap C)$
- L a comonad

Categorical structures for the QLC

A *concrete model of the QLC* [Selinger & Valiron '09] is:

$$\begin{tikzcd}
(\mathbf{C}, \otimes, I) \arrow[loop left, "L"] \arrow[loop right, "T"]
\end{tikzcd}$$

- $(\mathbf{C}, \otimes, I)$ a SMC with finite coproducts $(\oplus, 0)$
- T a strong monad with a Kleisli exponential $\multimap$ s.t. $\mathbf{C}(A \otimes B, TC) \cong \mathbf{C}(A, B \multimap C)$
  - cf. Moggi's computational lambda calculi
- L a comonad

Categorical structures for the QLC

A *concrete model of the QLC* [Selinger & Valiron '09] is:

$$\begin{tikzcd}(\mathbf{C}, \otimes, I) \arrow[loop left, "L"] \arrow[loop right, "T"]\end{tikzcd}$$

- $(\mathbf{C}, \otimes, I)$ a SMC with finite coproducts $(\oplus, 0)$
- T a strong monad with a Kleisli exponential $\multimap$ s.t. $\mathbf{C}(A \otimes B, TC) \cong \mathbf{C}(A, B \multimap C)$
  - cf. Moggi's computational lambda calculi
- L a linear exponential comonad

Categorical structures for the QLC

A *concrete model of the QLC* [Selinger & Valiron '09] is:

$$\begin{tikzcd}(\mathbf{C}, \otimes, I) \arrow[loop left, "L"] \arrow[loop right, "T"]\end{tikzcd}$$

- $(\mathbf{C}, \otimes, I)$ a SMC with finite coproducts $(\oplus, 0)$
- T a strong monad with a Kleisli exponential $\multimap$ s.t. $\mathbf{C}(A \otimes B, TC) \cong \mathbf{C}(A, B \multimap C)$
    - cf. Moggi's computational lambda calculi
- L a linear exponential comonad
    - a categorical model of the exponential modality "!"

Categorical structures for the QLC

A *concrete model of the QLC* [Selinger & Valiron '09] is:

$$\begin{tikzcd} (\mathbf{C}, \otimes, I) \arrow[loop left, "L"] \arrow[loop right, "T"] \end{tikzcd}$$

- $(\mathbf{C}, \otimes, I)$ a SMC with finite coproducts $(\oplus, 0)$
- T a strong monad with a Kleisli exponential $\multimap$ s.t. $\mathbf{C}(A \otimes B, TC) \cong \mathbf{C}(A, B \multimap C)$
    - cf. Moggi's computational lambda calculi
- L a linear exponential comonad
    - a categorical model of the exponential modality "!"
- $\mathcal{K}\ell(T)$ contains the category $\mathbf{Q}$ [Selinger '04]

Categorical structures for the QLC

A *concrete model of the QLC* [Selinger & Valiron '09] is:

$$\begin{tikzcd}(\mathbf{C}, \otimes, I) \arrow[loop left, "L"] \arrow[loop right, "T"]\end{tikzcd}$$

- $(\mathbf{C}, \otimes, I)$ a SMC with finite coproducts $(\oplus, 0)$
- T a strong monad with a Kleisli exponential $\multimap$ s.t. $\mathbf{C}(A \otimes B, TC) \cong \mathbf{C}(A, B \multimap C)$
  - cf. Moggi's computational lambda calculi
- L a linear exponential comonad
  - a categorical model of the exponential modality "!"
- $\mathcal{K}\ell(T)$ contains the category $\mathbf{Q}$ [Selinger '04]
  - Quantum operations between fin. dim. algebras

Categorical structures for the QLC

A *concrete model of the QLC* [Selinger & Valiron '09] is:

$$\begin{tikzcd} (\mathbf{C}, \otimes, I) \arrow[loop left, "L"] \arrow[loop right, "T"] \end{tikzcd}$$

- $(\mathbf{C}, \otimes, I)$ a SMC with finite coproducts $(\oplus, 0)$
- T a strong monad with a Kleisli exponential $\multimap$ s.t. $\mathbf{C}(A \otimes B, TC) \cong \mathbf{C}(A, B \multimap C)$
  - cf. Moggi's computational lambda calculi
- L a linear exponential comonad
  - a categorical model of the exponential modality "!"
- $\mathcal{K}\ell(T)$ contains the category $\mathbf{Q}$ [Selinger '04]
  - Quantum operations between fin. dim. algebras
  - To interpret quantum primitives

Categorical structures for the QLC

A *concrete model of the QLC* [Selinger & Valiron '09] is:

$$\begin{tikzcd} (\mathbf{C}, \otimes, I) \arrow[loop left, "L"] \arrow[loop right, "T"] \end{tikzcd}$$

- $(\mathbf{C}, \otimes, I)$ a SMC with finite coproducts $(\oplus, 0)$
- T a strong monad with a Kleisli exponential $\multimap$ s.t. $\mathbf{C}(A \otimes B, TC) \cong \mathbf{C}(A, B \multimap C)$
  - cf. Moggi's computational lambda calculi
- L a linear exponential comonad
  - a categorical model of the exponential modality "!"
- $\mathcal{K}\ell(T)$ contains the category $\mathbf{Q}$ [Selinger '04]
  - Quantum operations between fin. dim. algebras
  - To interpret quantum primitives
- and certain conditions (e.g. L preserves $\otimes$,$\oplus$)

Our model

$$\begin{tikzcd} \text{comonad} & (\mathbf{C}, \otimes, I) \arrow[loop left, "L"] \arrow[loop right, "T"] & \text{monad} \end{tikzcd}$$

Our model

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

Our model

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

(Opposite categories of)

- $\mathbf{vN}$: v.N. algebras and normal unital $*$-homomorphisms (aka. normal MIU-maps)
- $\mathbf{vN}_{\mathrm{CPsU}}$: v.N. algebras and normal **c**ompletely **p**ositive **s**ub**u**ntial (CPsU) maps

Our model

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}_{\mathrm{CPsU}}^{\mathrm{op}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

(Opposite categories of)

structure-preserving maps

- $\mathbf{vN}$: v.N. algebras and normal unital $*$-homomorphisms (aka. normal MIU-maps)
- $\mathbf{vN}_{\mathrm{CPsU}}$: v.N. algebras and normal **c**ompletely **p**ositive **s**ub**u**ntial (CPsU) maps

Our model

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

(Opposite categories of)

structure-preserving maps

- $\mathbf{vN}$: v.N. algebras and normal unital $*$-homomorphisms (aka. normal MIU-maps)
- $\mathbf{vN}_{\mathrm{CPsU}}$: v.N. algebras and normal **c**ompletely **p**ositive **s**ub**u**ntial (CPsU) maps

quantum processes/operations

Our model

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

(Opposite categories of)

- $\mathbf{vN}$: v.N. algebras and normal unital $*$-homomorphisms (aka. normal MIU-maps)

structure-preserving maps

- $\mathbf{vN}_{\mathrm{CPsU}}$: v.N. algebras and normal **c**ompletely **p**ositive **s**ub**u**ntial (CPsU) maps

quantum processes/operations

- $\mathbf{vN} \subseteq \mathbf{vN}_{\mathrm{CPsU}}$

Our model

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^{\infty}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

(Opposite categories of)

structure-preserving maps

- $\mathbf{vN}$: v.N. algebras and normal unital $*$-homomorphisms (aka. normal MIU-maps)
- $\mathbf{vN}_{\mathrm{CPsU}}$: v.N. algebras and normal **c**ompletely **p**ositive **s**ub**u**ntial (CPsU) maps
- $\mathbf{vN} \subseteq \mathbf{vN}_{\mathrm{CPsU}}$

quantum processes/operations

Goal. $\mathbf{vN}^{\mathrm{op}}$ forms a concrete model of the QLC.

Our model

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

(Opposite categories of)

structure-preserving maps

- $\mathbf{vN}$: v.N. algebras and normal unital $*$-homomorphisms (aka. normal MIU-maps)
- $\mathbf{vN}_{\mathrm{CPsU}}$: v.N. algebras and normal **c**ompletely **p**ositive **s**ub**u**ntial (CPsU) maps
- $\mathbf{vN} \subseteq \mathbf{vN}_{\mathrm{CPsU}}$

quantum processes/operations

Goal. $\mathbf{vN}^{\mathrm{op}}$ forms a concrete model of the QLC.

① $\mathbf{vN}^{\mathrm{op}}$ is symm. mon. via tensor products $\otimes$ and $\mathbb{C}$

Our model

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

(Opposite categories of)

structure-preserving maps

- $\mathbf{vN}$: v.N. algebras and normal unital $*$-homomorphisms (aka. normal MIU-maps)
- $\mathbf{vN}_{\mathrm{CPsU}}$: v.N. algebras and normal **c**ompletely **p**ositive **s**ub**u**ntial (CPsU) maps
- $\mathbf{vN} \subseteq \mathbf{vN}_{\mathrm{CPsU}}$

quantum processes/operations

Goal. $\mathbf{vN}^{\mathrm{op}}$ forms a concrete model of the QLC.

① $\mathbf{vN}^{\mathrm{op}}$ is symm. mon. via tensor products $\otimes$ and $\mathbb{C}$

② $\mathbf{vN}^{\mathrm{op}}$ has coproducts given by direct sums $\oplus$

Our model

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^{\infty}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

(Opposite categories of)

structure-preserving maps

- $\mathbf{vN}$: v.N. algebras and normal unital $*$-homomorphisms (aka. normal MIU-maps)
- $\mathbf{vN}_{\mathrm{CPsU}}$: v.N. algebras and normal **c**ompletely **p**ositive **s**ub**u**ntial (CPsU) maps
- $\mathbf{vN} \subseteq \mathbf{vN}_{\mathrm{CPsU}}$

quantum processes/operations

Goal. $\mathbf{vN}^{\mathrm{op}}$ forms a concrete model of the QLC.

① $\mathbf{vN}^{\mathrm{op}}$ is symm. mon. via tensor products $\otimes$ and $\mathbb{C}$

② $\mathbf{vN}^{\mathrm{op}}$ has coproducts given by direct sums $\oplus$

products in $\mathbf{vN}$

A result of Andre Kornell

Theorem (Kornell 2012)**.** *The SMC* $(\mathbf{vN}^{\mathrm{op}}, \otimes, \mathbb{C})$ *is* ***closed****. Namely: for any v.N. alg.* $\mathscr{A}, \mathscr{B}$ *there is* $\mathscr{B}^{*\mathscr{A}}$ *(called the free exponential) s.t.*

$$\mathbf{vN}^{\mathrm{op}}(\mathscr{C} \otimes \mathscr{A}, \mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}(\mathscr{C}, \mathscr{B}^{*\mathscr{A}})$$

A result of Andre Kornell

Theorem (Kornell 2012)**.** *The SMC* $(\mathbf{vN}^{\mathrm{op}}, \otimes, \mathbb{C})$ *is* ***closed****. Namely: for any v.N. alg.* $\mathscr{A}, \mathscr{B}$ *there is* $\mathscr{B}^{*\mathscr{A}}$ *(called the free exponential) s.t.*

$$\mathbf{vN}^{\mathrm{op}}(\mathscr{C} \otimes \mathscr{A}, \mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}(\mathscr{C}, \mathscr{B}^{*\mathscr{A}})$$

- Appeared (only) at arXiv:1202.2994 [math.OA]

A result of Andre Kornell

Theorem (Kornell 2012)**.** *The SMC* $(\mathbf{vN}^{\mathrm{op}}, \otimes, \mathbb{C})$ *is **closed**. Namely: for any v.N. alg.* $\mathscr{A}, \mathscr{B}$ *there is* $\mathscr{B}^{*\mathscr{A}}$ *(called the free exponential) s.t.*

$$\mathbf{vN}^{\mathrm{op}}(\mathscr{C} \otimes \mathscr{A}, \mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}(\mathscr{C}, \mathscr{B}^{*\mathscr{A}})$$

- Appeared (only) at arXiv:1202.2994 [math.OA]

Alternative proof by applying Adjoint Functor Theorem to $(-) \otimes \mathscr{A} : \mathbf{vN} \to \mathbf{vN}$

A result of Andre Kornell

Theorem (Kornell 2012). *The SMC* $(\mathbf{vN}^{\mathrm{op}}, \otimes, \mathbb{C})$ *is* ***closed****. Namely: for any v.N. alg.* $\mathscr{A}, \mathscr{B}$ *there is* $\mathscr{B}^{*\mathscr{A}}$ *(called the free exponential) s.t.*

$$\mathbf{vN}^{\mathrm{op}}(\mathscr{C} \otimes \mathscr{A}, \mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}(\mathscr{C}, \mathscr{B}^{*\mathscr{A}})$$

- Appeared (only) at arXiv:1202.2994 [math.OA]

Alternative proof by applying Adjoint Functor Theorem to $(-) \otimes \mathscr{A} : \mathbf{vN} \to \mathbf{vN}$

① $\mathbf{vN}$ is complete, locally small

A result of Andre Kornell

Theorem (Kornell 2012)**.** *The SMC* $(\mathbf{vN}^{\mathrm{op}}, \otimes, \mathbb{C})$ *is* ***closed****. Namely: for any v.N. alg.* $\mathscr{A}, \mathscr{B}$ *there is* $\mathscr{B}^{*\mathscr{A}}$ *(called the free exponential) s.t.*

$$\mathbf{vN}^{\mathrm{op}}(\mathscr{C} \otimes \mathscr{A}, \mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}(\mathscr{C}, \mathscr{B}^{*\mathscr{A}})$$

- Appeared (only) at arXiv:1202.2994 [math.OA]

Alternative proof by applying Adjoint Functor Theorem to $(-) \otimes \mathscr{A} : \mathbf{vN} \to \mathbf{vN}$

1. $\mathbf{vN}$ is complete, locally small
2. $(-) \otimes \mathscr{A}$ preserves limits

A result of Andre Kornell

Theorem (Kornell 2012)**.** *The SMC* $(\mathbf{vN}^{\mathrm{op}}, \otimes, \mathbb{C})$ *is* ***closed****. Namely: for any v.N. alg.* $\mathscr{A}, \mathscr{B}$ *there is* $\mathscr{B}^{*\mathscr{A}}$ *(called the free exponential) s.t.*

$$\mathbf{vN}^{\mathrm{op}}(\mathscr{C} \otimes \mathscr{A}, \mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}(\mathscr{C}, \mathscr{B}^{*\mathscr{A}})$$

- Appeared (only) at arXiv:1202.2994 [math.OA]

Alternative proof by applying Adjoint Functor Theorem to $(-) \otimes \mathscr{A} : \mathbf{vN} \to \mathbf{vN}$

1. $\mathbf{vN}$ is complete, locally small
2. $(-) \otimes \mathscr{A}$ preserves limits

$(\bigoplus_i \mathscr{B}_i) \otimes \mathscr{A} \cong \bigoplus_i \mathscr{B}_i \otimes \mathscr{A}$

A result of Andre Kornell

Theorem (Kornell 2012). *The SMC $(\mathbf{vN}^{\mathrm{op}}, \otimes, \mathbb{C})$ is* ***closed****. Namely: for any v.N. alg. $\mathscr{A}, \mathscr{B}$ there is $\mathscr{B}^{*\mathscr{A}}$ (called the free exponential) s.t.*

$$\mathbf{vN}^{\mathrm{op}}(\mathscr{C} \otimes \mathscr{A}, \mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}(\mathscr{C}, \mathscr{B}^{*\mathscr{A}})$$

- Appeared (only) at arXiv:1202.2994 [math.OA]

Alternative proof by applying Adjoint Functor Theorem to $(-) \otimes \mathscr{A} : \mathbf{vN} \to \mathbf{vN}$

1. $\mathbf{vN}$ is complete, locally small
2. $(-) \otimes \mathscr{A}$ preserves limits
3. Solution Set Condition

$(\bigoplus_i \mathscr{B}_i) \otimes \mathscr{A} \cong \bigoplus_i \mathscr{B}_i \otimes \mathscr{A}$

A result of Andre Kornell

Theorem (Kornell 2012). *The SMC* $(\mathbf{vN}^{\mathrm{op}}, \otimes, \mathbb{C})$ *is* ***closed***. *Namely: for any v.N. alg.* $\mathscr{A}, \mathscr{B}$ *there is* $\mathscr{B}^{*\mathscr{A}}$ *(called the free exponential) s.t.*

$$\mathbf{vN}^{\mathrm{op}}(\mathscr{C} \otimes \mathscr{A}, \mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}(\mathscr{C}, \mathscr{B}^{*\mathscr{A}})$$

- Appeared (only) at arXiv:1202.2994 [math.OA]

Alternative proof by applying Adjoint Functor Theorem to $(-) \otimes \mathscr{A} : \mathbf{vN} \to \mathbf{vN}$

1. $\mathbf{vN}$ is complete, locally small
2. $(-) \otimes \mathscr{A}$ preserves limits
3. Solution Set Condition

$(\bigoplus_i \mathscr{B}_i) \otimes \mathscr{A} \cong \bigoplus_i \mathscr{B}_i \otimes \mathscr{A}$

Warning: we do not know a good description of the free exponential. (Even ${\mathcal{M}_2}^{*\mathcal{M}_2}$ is hard!)

Monad part (right-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^{\infty}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

Monad part (right-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

① **The previous talk** by A.W.:

- The inclusion $\mathcal{J}$ has a right adjoint $\mathcal{F}$ (via AFT)
- $\mathcal{K}\ell(\mathcal{F}\mathcal{J}) \cong \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$
  (since $\mathbf{vN}^{\mathrm{op}}(\mathscr{A}, \mathcal{F}\mathcal{J}\mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}(\mathscr{A}, \mathscr{B})$)

Monad part (right-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

1. **The previous talk** by A.W.:
    - The inclusion $\mathcal{J}$ has a right adjoint $\mathcal{F}$ (via AFT)
    - $\mathcal{K}\ell(\mathcal{F}\mathcal{J}) \cong \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$
      (since $\mathbf{vN}^{\mathrm{op}}(\mathscr{A}, \mathcal{F}\mathcal{J}\mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}(\mathscr{A}, \mathscr{B})$)
2. $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$ contains $\mathbf{Q}$ (in fact, $\mathbf{fdvN}^{\mathrm{op}}_{\mathrm{CPsU}} \simeq \mathbf{Q}$)

Monad part (right-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

① **The previous talk** by A.W.:

- The inclusion $\mathcal{J}$ has a right adjoint $\mathcal{F}$ (via AFT)
- $\mathcal{K}\ell(\mathcal{F}\mathcal{J}) \cong \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$
  (since $\mathbf{vN}^{\mathrm{op}}(\mathscr{A}, \mathcal{F}\mathcal{J}\mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}(\mathscr{A}, \mathscr{B})$)

② $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$ contains $\mathbf{Q}$ (in fact, $\mathbf{fdvN}^{\mathrm{op}}_{\mathrm{CPsU}} \simeq \mathbf{Q}$)

③ $\mathcal{F}\mathcal{J}$ is a strong monad, since $\mathcal{J}$ is strict monoidal

Monad part (right-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

1. **The previous talk** by A.W.:
   - The inclusion $\mathcal{J}$ has a right adjoint $\mathcal{F}$ (via AFT)
   - $\mathcal{K}\ell(\mathcal{F}\mathcal{J}) \cong \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$
     (since $\mathbf{vN}^{\mathrm{op}}(\mathscr{A}, \mathcal{F}\mathcal{J}\mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}(\mathscr{A}, \mathscr{B})$)
2. $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$ contains $\mathbf{Q}$ (in fact, $\mathbf{fdvN}^{\mathrm{op}}_{\mathrm{CPsU}} \simeq \mathbf{Q}$)
3. $\mathcal{F}\mathcal{J}$ is a strong monad, since $\mathcal{J}$ is strict monoidal
4. Kleisli exponential $\mathscr{A} \multimap \mathscr{B} := (\mathcal{F}\mathcal{J}\mathscr{B})^{*\mathscr{A}}$

Monad part (right-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

① **The previous talk** by A.W.:

- The inclusion $\mathcal{J}$ has a right adjoint $\mathcal{F}$ (via AFT)
- $\mathcal{K}\ell(\mathcal{F}\mathcal{J}) \cong \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$
  (since $\mathbf{vN}^{\mathrm{op}}(\mathscr{A}, \mathcal{F}\mathcal{J}\mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}(\mathscr{A}, \mathscr{B})$)

② $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$ contains $\mathbf{Q}$ (in fact, $\mathbf{fdvN}^{\mathrm{op}}_{\mathrm{CPsU}} \simeq \mathbf{Q}$)

③ $\mathcal{F}\mathcal{J}$ is a strong monad, since $\mathcal{J}$ is strict monoidal

④ Kleisli exponential $\mathscr{A} \multimap \mathscr{B} := (\mathcal{F}\mathcal{J}\mathscr{B})^{*\mathscr{A}}$

- $\mathbf{vN}^{\mathrm{op}}(\mathscr{C} \otimes \mathscr{A}, \mathcal{F}\mathcal{J}\mathscr{B}) \cong \mathbf{vN}^{\mathrm{op}}(\mathscr{C}, (\mathcal{F}\mathcal{J}\mathscr{B})^{*\mathscr{A}})$

Linear exponential comonads

Linear exponential comonads

$=$ Categorical models of the exponential modality "!"

Linear exponential comonads

$=$ Categorical models of the exponential modality "!"

A comonad L is *linear exponential* when endowed with a comonoid structure on each object LA:

$$LA \longrightarrow LA \otimes LA \qquad LA \longrightarrow I$$

(suitably compatible with the comonad structures)

Linear exponential comonads

$=$ Categorical models of the exponential modality "!"

A comonad L is *linear exponential* when endowed with a comonoid structure on each object LA:

Contraction (Duplication)

$$LA \longrightarrow LA \otimes LA \qquad LA \longrightarrow I$$

Weakening (Discarding)

(suitably compatible with the comonad structures)

Linear exponential comonads

$=$ Categorical models of the exponential modality "!"

A comonad L is *linear exponential* when endowed with a comonoid structure on each object LA:

Contraction (Duplication)

$$LA \longrightarrow LA \otimes LA \qquad LA \longrightarrow I$$

Weakening (Discarding)

(suitably compatible with the comonad structures)

Theorem (Benton)**.** *If we have a symm. mon. adjunction between a SMC and a cartesian monoidal category as in*

$$\begin{tikzcd}(\mathbf{B}, \times, 1) \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & (\mathbf{C}, \otimes, I) \arrow[l, bend left, "G"]\end{tikzcd}$$

then the comonad FG on $\mathbf{C}$ is linear exponential.

Comonad part (left-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^{\infty}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

Comonad part (left-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

① $\mathbf{vN}(-, \mathbb{C})$ is a hom-functor

Comonad part (left-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

1. $\mathbf{vN}(-, \mathbb{C})$ is a hom-functor
2. For each set X, $\ell^\infty(X) = \{\text{bounded } \varphi \colon X \to \mathbb{C}\}$ is a v.N. algebra, giving a functor ℓ^∞

Comonad part (left-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^{\infty}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

1. $\mathbf{vN}(-, \mathbb{C})$ is a hom-functor
2. For each set X, $\ell^{\infty}(X) = \{\text{bounded } \varphi \colon X \to \mathbb{C}\}$ is a v.N. algebra, giving a functor ℓ^{∞}
3. The dual adjunction $\mathbf{Set} \rightleftarrows \mathbf{vN}^{\mathrm{op}}$ via "swapping arguments" $f(x)(a) = g(a)(x)$ for $f \colon X \to \mathbf{vN}(\mathscr{A}, \mathbb{C})$ and $g \colon \mathscr{A} \to \ell^{\infty}(X)$

Comonad part (left-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

① $\mathbf{vN}(-,\mathbb{C})$ is a hom-functor

② For each set X, $\ell^\infty(X) = \{\text{bounded } \varphi\colon X \to \mathbb{C}\}$ is a v.N. algebra, giving a functor ℓ^∞

③ The dual adjunction $\mathbf{Set} \rightleftarrows \mathbf{vN}^{\mathrm{op}}$ via "swapping arguments" $f(x)(a) = g(a)(x)$ for $f\colon X \to \mathbf{vN}(\mathscr{A},\mathbb{C})$ and $g\colon \mathscr{A} \to \ell^\infty(X)$

④ $\mathbf{Set}$ is cartesian $(\times, 1)$

Comonad part (left-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

① $\mathbf{vN}(-,\mathbb{C})$ is a hom-functor

② For each set X, $\ell^\infty(X) = \{\text{bounded } \varphi\colon X \to \mathbb{C}\}$ is a v.N. algebra, giving a functor ℓ^∞

③ The dual adjunction $\mathbf{Set} \rightleftarrows \mathbf{vN}^{\mathrm{op}}$ via "swapping arguments" $f(x)(a) = g(a)(x)$ for $f\colon X \to \mathbf{vN}(\mathscr{A},\mathbb{C})$ and $g\colon \mathscr{A} \to \ell^\infty(X)$

④ $\mathbf{Set}$ is cartesian $(\times, 1)$

⑤ $\mathbf{Set} \rightleftarrows \mathbf{vN}^{\mathrm{op}}$ is monoidal: $\ell^\infty(X \times Y) \cong \ell^\infty(X) \otimes \ell^\infty(Y)$

Comonad part (left-hand side)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

① $\mathbf{vN}(-,\mathbb{C})$ is a hom-functor

② For each set X, $\ell^\infty(X) = \{$bounded $\varphi\colon X \to \mathbb{C}\}$ is a v.N. algebra, giving a functor ℓ^∞

③ The dual adjunction $\mathbf{Set} \rightleftarrows \mathbf{vN}^{\mathrm{op}}$ via "swapping arguments" $f(x)(a) = g(a)(x)$ for $f\colon X \to \mathbf{vN}(\mathscr{A},\mathbb{C})$ and $g\colon \mathscr{A} \to \ell^\infty(X)$

④ $\mathbf{Set}$ is cartesian $(\times, 1)$

⑤ $\mathbf{Set} \rightleftarrows \mathbf{vN}^{\mathrm{op}}$ is monoidal: $\ell^\infty(X \times Y) \cong \ell^\infty(X) \otimes \ell^\infty(Y)$

⑥ $\ell^\infty(\mathbf{vN}(-,\mathbb{C}))$ is linear exponential by Benton

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

forms a concrete model of the QLC, in the sense of S.&V.

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

forms a concrete model of the QLC, in the sense of S.&V.

Interpretation of types

$$\llbracket \top \rrbracket = \mathbb{C} \qquad \llbracket \mathtt{qbit} \rrbracket = \mathcal{M}_2$$

$$\llbracket A \otimes B \rrbracket = \llbracket A \rrbracket \otimes \llbracket B \rrbracket \qquad \llbracket A \oplus B \rrbracket = \llbracket A \rrbracket \oplus \llbracket B \rrbracket$$

$$\llbracket A \multimap B \rrbracket = (\mathcal{F}\mathcal{J}\llbracket B \rrbracket)^{*\llbracket A \rrbracket} \qquad \llbracket !A \rrbracket = \ell^\infty(\mathbf{vN}(\llbracket A \rrbracket, \mathbb{C}))$$

$$\begin{tikzcd}\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]\end{tikzcd}$$

forms a concrete model of the QLC, in the sense of S.&V.

Interpretation of types

$$\llbracket \top \rrbracket = \mathbb{C} \qquad \llbracket \mathtt{qbit} \rrbracket = \mathcal{M}_2$$

$$\llbracket A \otimes B \rrbracket = \llbracket A \rrbracket \otimes \llbracket B \rrbracket \qquad \llbracket A \oplus B \rrbracket = \llbracket A \rrbracket \oplus \llbracket B \rrbracket$$

$$\llbracket A \multimap B \rrbracket = (\mathcal{F}\mathcal{J}\llbracket B \rrbracket)^{*\llbracket A \rrbracket} \qquad \llbracket !A \rrbracket = \ell^\infty(\mathbf{vN}(\llbracket A \rrbracket, \mathbb{C}))$$

Interpretation of terms

Well-typed term $x : A \vdash M : B$ is interpreted by

- a Kleisli map $\llbracket A \rrbracket \xrightarrow{\llbracket M \rrbracket} \mathcal{F}\mathcal{J}\llbracket B \rrbracket$ in $\mathbf{vN}^{\mathrm{op}}$

$$\begin{tikzcd}\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]\end{tikzcd}$$

forms a concrete model of the QLC, in the sense of S.&V.

Interpretation of types

$$[\![\top]\!] = \mathbb{C} \qquad [\![\mathtt{qbit}]\!] = \mathcal{M}_2$$

$$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!] \qquad [\![A \oplus B]\!] = [\![A]\!] \oplus [\![B]\!]$$

$$[\![A \multimap B]\!] = (\mathcal{F}\mathcal{J}[\![B]\!])^{*[\![A]\!]} \qquad [\![!A]\!] = \ell^\infty(\mathbf{vN}([\![A]\!], \mathbb{C}))$$

Interpretation of terms

Well-typed term $x : A \vdash M : B$ is interpreted by

- a Kleisli map $[\![A]\!] \xrightarrow{[\![M]\!]} \mathcal{F}\mathcal{J}[\![B]\!]$ in $\mathbf{vN}^{\mathrm{op}}$
- *i.e.* a map $[\![A]\!] \to [\![B]\!]$ in $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

forms a concrete model of the QLC, in the sense of S.&V.

Interpretation of types

$$[\![\top]\!] = \mathbb{C} \qquad [\![\mathtt{qbit}]\!] = \mathcal{M}_2$$

$$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!] \qquad [\![A \oplus B]\!] = [\![A]\!] \oplus [\![B]\!]$$

$$[\![A \multimap B]\!] = (\mathcal{F}\mathcal{J}[\![B]\!])^{*[\![A]\!]} \qquad [\![!A]\!] = \ell^\infty(\mathbf{vN}([\![A]\!], \mathbb{C}))$$

Interpretation of terms

Well-typed term $x : A \vdash M : B$ is interpreted by

- a Kleisli map $[\![A]\!] \xrightarrow{[\![M]\!]} \mathcal{F}\mathcal{J}[\![B]\!]$ in $\mathbf{vN}^{\mathrm{op}}$
- *i.e.* a map $[\![A]\!] \to [\![B]\!]$ in $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$
- *i.e.* a normal CPsU-map $[\![B]\!] \to [\![A]\!]$ (quantum process!)

Examples of interpretations

$$\llbracket !\top \rrbracket \cong \llbracket \top \rrbracket = \mathbb{C}$$

Examples of interpretations

$$\llbracket !\top \rrbracket \cong \llbracket \top \rrbracket = \mathbb{C}$$

$$\llbracket !\mathtt{bit} \rrbracket \cong \llbracket \mathtt{bit} \rrbracket = \mathbb{C}^2 \qquad (\mathtt{bit} \coloneqq \top \oplus \top)$$

Examples of interpretations

$$\begin{aligned}
[\![!\top]\!] &\cong [\![\top]\!] = \mathbb{C} \\
[\![!\mathtt{bit}]\!] &\cong [\![\mathtt{bit}]\!] = \mathbb{C}^2 \qquad (\mathtt{bit} := \top \oplus \top) \\
[\![!\mathtt{qbit}]\!] &= \ell^\infty(\mathbf{vN}(\mathcal{M}_2, \mathbb{C})) \cong \{0\}
\end{aligned}$$

since $\mathbf{vN}(\mathcal{M}_2, \mathbb{C}) = \varnothing$.

Examples of interpretations

$$\llbracket !\top \rrbracket \cong \llbracket \top \rrbracket = \mathbb{C}$$

$$\llbracket !\mathtt{bit} \rrbracket \cong \llbracket \mathtt{bit} \rrbracket = \mathbb{C}^2 \qquad (\mathtt{bit} := \top \oplus \top)$$

$$\llbracket !\mathtt{qbit} \rrbracket = \ell^\infty(\mathbf{vN}(\mathcal{M}_2, \mathbb{C})) \cong \{0\}$$

initial in $\mathbf{vN}^{\mathrm{op}}$

since $\mathbf{vN}(\mathcal{M}_2, \mathbb{C}) = \varnothing$. This is as expected, because there is no valid typing judgement $\vdash M : !\mathtt{qbit}$.

Examples of interpretations

$$\llbracket !\top \rrbracket \cong \llbracket \top \rrbracket = \mathbb{C}$$

$$\llbracket !\mathtt{bit} \rrbracket \cong \llbracket \mathtt{bit} \rrbracket = \mathbb{C}^2 \qquad (\mathtt{bit} := \top \oplus \top)$$

$$\llbracket !\mathtt{qbit} \rrbracket = \ell^\infty(\mathbf{vN}(\mathcal{M}_2, \mathbb{C})) \cong \{0\}$$

initial in $\mathbf{vN}^{\mathrm{op}}$

since $\mathbf{vN}(\mathcal{M}_2, \mathbb{C}) = \varnothing$. This is as expected, because there is no valid typing judgement $\vdash M : !\mathtt{qbit}$.

$$\llbracket A \multimap B \rrbracket = (\mathcal{F}\mathcal{J}\llbracket B \rrbracket)^{*\llbracket A \rrbracket}$$

Examples of interpretations

$$\llbracket !\top \rrbracket \cong \llbracket \top \rrbracket = \mathbb{C}$$

$$\llbracket !\mathtt{bit} \rrbracket \cong \llbracket \mathtt{bit} \rrbracket = \mathbb{C}^2 \qquad (\mathtt{bit} := \top \oplus \top)$$

$$\llbracket !\mathtt{qbit} \rrbracket = \ell^\infty(\mathbf{vN}(\mathcal{M}_2, \mathbb{C})) \cong \{0\}$$

initial in $\mathbf{vN}^{\mathrm{op}}$

since $\mathbf{vN}(\mathcal{M}_2, \mathbb{C}) = \varnothing$. This is as expected, because there is no valid typing judgement $\vdash M : !\mathtt{qbit}$.

$$\llbracket A \multimap B \rrbracket = (\mathcal{F}\mathcal{J}\llbracket B \rrbracket)^{*\llbracket A \rrbracket} = ??$$

Examples of interpretations

$$\begin{aligned}
[\![!\top]\!] &\cong [\![\top]\!] = \mathbb{C} \\
[\![!\mathtt{bit}]\!] &\cong [\![\mathtt{bit}]\!] = \mathbb{C}^2 \qquad (\mathtt{bit} := \top \oplus \top) \\
[\![!\mathtt{qbit}]\!] &= \ell^\infty(\mathbf{vN}(\mathcal{M}_2, \mathbb{C})) \cong \{0\}
\end{aligned}$$

initial in $\mathbf{vN}^{\mathrm{op}}$

since $\mathbf{vN}(\mathcal{M}_2, \mathbb{C}) = \varnothing$. This is as expected, because there is no valid typing judgement $\vdash M : !\mathtt{qbit}$.

$$\begin{aligned}
[\![A \multimap B]\!] &= (\mathcal{F}\mathcal{J}[\![B]\!])^{*[\![A]\!]} = \textcolor{red}{??} \\
[\![!(A \multimap B)]\!] &= \ell^\infty(\mathbf{vN}((\mathcal{F}\mathcal{J}[\![B]\!])^{*[\![A]\!]}, \mathbb{C}))
\end{aligned}$$

Examples of interpretations

$$\llbracket !\top \rrbracket \cong \llbracket \top \rrbracket = \mathbb{C}$$

$$\llbracket !\mathtt{bit} \rrbracket \cong \llbracket \mathtt{bit} \rrbracket = \mathbb{C}^2 \qquad (\mathtt{bit} := \top \oplus \top)$$

$$\llbracket !\mathtt{qbit} \rrbracket = \ell^\infty(\mathbf{vN}(\mathcal{M}_2, \mathbb{C})) \cong \{0\}$$

initial in $\mathbf{vN}^{\mathrm{op}}$

since $\mathbf{vN}(\mathcal{M}_2, \mathbb{C}) = \varnothing$. This is as expected, because there is no valid typing judgement $\vdash M : !\mathtt{qbit}$.

$$\llbracket A \multimap B \rrbracket = (\mathcal{F}\mathcal{J}\llbracket B \rrbracket)^{*\llbracket A \rrbracket} = ??$$

$$\begin{aligned} \llbracket !(A \multimap B) \rrbracket &= \ell^\infty(\mathbf{vN}((\mathcal{F}\mathcal{J}\llbracket B \rrbracket)^{*\llbracket A \rrbracket}, \mathbb{C})) \\ &\cong \ell^\infty(\mathbf{vN}(\mathcal{F}\mathcal{J}\llbracket B \rrbracket, \llbracket A \rrbracket)) \end{aligned}$$

Examples of interpretations

$$\begin{aligned}
[\![!\top]\!] &\cong [\![\top]\!] = \mathbb{C} \\
[\![!\mathtt{bit}]\!] &\cong [\![\mathtt{bit}]\!] = \mathbb{C}^2 \qquad (\mathtt{bit} := \top \oplus \top) \\
[\![!\mathtt{qbit}]\!] &= \ell^\infty(\mathbf{vN}(\mathcal{M}_2, \mathbb{C})) \cong \{0\}
\end{aligned}$$

initial in $\mathbf{vN}^{\mathrm{op}}$

since $\mathbf{vN}(\mathcal{M}_2, \mathbb{C}) = \varnothing$. This is as expected, because there is no valid typing judgement $\vdash M : !\mathtt{qbit}$.

$$\begin{aligned}
[\![A \multimap B]\!] &= (\mathcal{F}\mathcal{J}[\![B]\!])^{*[\![A]\!]} = ?? \\
[\![!(A \multimap B)]\!] &= \ell^\infty(\mathbf{vN}((\mathcal{F}\mathcal{J}[\![B]\!])^{*[\![A]\!]}, \mathbb{C})) \\
&\cong \ell^\infty(\mathbf{vN}(\mathcal{F}\mathcal{J}[\![B]\!], [\![A]\!])) \\
&\cong \ell^\infty(\mathbf{vN}_{\mathrm{CPsU}}([\![B]\!], [\![A]\!]))
\end{aligned}$$

Examples of interpretations

$$\begin{aligned} [\![!\top]\!] &\cong [\![\top]\!] = \mathbb{C} \\ [\![!\mathtt{bit}]\!] &\cong [\![\mathtt{bit}]\!] = \mathbb{C}^2 \qquad (\mathtt{bit} := \top \oplus \top) \\ [\![!\mathtt{qbit}]\!] &= \ell^\infty(\mathbf{vN}(\mathcal{M}_2, \mathbb{C})) \cong \{0\} \end{aligned}$$

initial in $\mathbf{vN}^{\mathrm{op}}$

since $\mathbf{vN}(\mathcal{M}_2, \mathbb{C}) = \varnothing$. This is as expected, because there is no valid typing judgement $\vdash M : !\mathtt{qbit}$.

$$\begin{aligned} [\![A \multimap B]\!] &= (\mathcal{F}\mathcal{J}[\![B]\!])^{*[\![A]\!]} = ?? \\ [\![!(A \multimap B)]\!] &= \ell^\infty(\mathbf{vN}((\mathcal{F}\mathcal{J}[\![B]\!])^{*[\![A]\!]}, \mathbb{C})) \\ &\cong \ell^\infty(\mathbf{vN}(\mathcal{F}\mathcal{J}[\![B]\!], [\![A]\!])) \\ &\cong \ell^\infty(\mathbf{vN}_{\mathrm{CPsU}}([\![B]\!], [\![A]\!])) \end{aligned}$$

quantum processes

Remarks

- Duplicable types $!A$ are interpreted by $\ell^\infty(X)$, rather than arbitrary commutative von Neumann algebras such as $L^\infty(X, \mu)$

Remarks

- Duplicable types $!A$ are interpreted by $\ell^{\infty}(X)$, rather than arbitrary commutative von Neumann algebras such as $L^{\infty}(X,\mu)$
  - There exists an adjunction $\mathbf{CvN}^{\mathrm{op}} \leftrightarrows \mathbf{vN}^{\mathrm{op}}$, which does not give a linear exponential comonad

Remarks

- Duplicable types $!A$ are interpreted by $\ell^{\infty}(X)$, rather than arbitrary commutative von Neumann algebras such as $L^{\infty}(X, \mu)$
  - There exists an adjunction $\mathbf{CvN}^{\mathrm{op}} \leftrightarrows \mathbf{vN}^{\mathrm{op}}$, which does not give a linear exponential comonad
  - In fact, any comonoid in the SMC $\mathbf{vN}^{\mathrm{op}}$ (or $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$) must be of the form $\ell^{\infty}(X)$

Remarks

- Duplicable types $!A$ are interpreted by $\ell^\infty(X)$, rather than arbitrary commutative von Neumann algebras such as $L^\infty(X, \mu)$
  - There exists an adjunction $\mathbf{CvN}^{\mathrm{op}} \leftrightarrows \mathbf{vN}^{\mathrm{op}}$, which does not give a linear exponential comonad
  - In fact, any comonoid in the SMC $\mathbf{vN}^{\mathrm{op}}$ (or $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$) must be of the form $\ell^\infty(X)$
- C^*-algebras do not work similarly, since $\mathbf{Cstar}^{\mathrm{op}}$ is not a closed SMC

Remarks

- Duplicable types $!A$ are interpreted by $\ell^\infty(X)$, rather than arbitrary commutative von Neumann algebras such as $L^\infty(X, \mu)$
  - There exists an adjunction $\mathbf{CvN}^{\mathrm{op}} \leftrightarrows \mathbf{vN}^{\mathrm{op}}$, which does not give a linear exponential comonad
  - In fact, any comonoid in the SMC $\mathbf{vN}^{\mathrm{op}}$ (or $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$) must be of the form $\ell^\infty(X)$
- C^*-algebras do not work similarly, since $\mathbf{Cstar}^{\mathrm{op}}$ is not a closed SMC
  - $\otimes$ does not distribute over infinite $\bigoplus$

Remarks

- Duplicable types $!A$ are interpreted by $\ell^\infty(X)$, rather than arbitrary commutative von Neumann algebras such as $L^\infty(X, \mu)$
  - There exists an adjunction $\mathbf{CvN}^{\mathrm{op}} \leftrightarrows \mathbf{vN}^{\mathrm{op}}$, which does not give a linear exponential comonad
  - In fact, any comonoid in the SMC $\mathbf{vN}^{\mathrm{op}}$ (or $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$) must be of the form $\ell^\infty(X)$
- C^*-algebras do not work similarly, since $\mathbf{Cstar}^{\mathrm{op}}$ is not a closed SMC
  - $\otimes$ does not distribute over infinite $\bigoplus$
- Our model is adequate wrt. the operational semantics

Remarks

- Duplicable types $!A$ are interpreted by $\ell^\infty(X)$, rather than arbitrary commutative von Neumann algebras such as $L^\infty(X,\mu)$
  - There exists an adjunction $\mathbf{CvN}^{\mathrm{op}} \leftrightarrows \mathbf{vN}^{\mathrm{op}}$, which does not give a linear exponential comonad
  - In fact, any comonoid in the SMC $\mathbf{vN}^{\mathrm{op}}$ (or $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$) must be of the form $\ell^\infty(X)$
- C^*-algebras do not work similarly, since $\mathbf{Cstar}^{\mathrm{op}}$ is not a closed SMC
  - $\otimes$ does not distribute over infinite $\bigoplus$
- Our model is adequate wrt. the operational semantics
  - Laborious but straightforward, since our language does not contain recursion

Conclusions

Von Neumann algebras are powerful enough to interpret Selinger & Valiron's Quantum Lambda Calculus, via the adjunctions:

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, shift left=1.5, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, shift left=1.5, "\mathbf{vN}(-{,}\mathbb{C})"] \arrow[r, hook, shift left=1.5, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, shift left=1.5, "\mathcal{F}"]
\end{tikzcd}$$

Future work:

- Recursion
    - $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$ is dcpo-enriched, but $\mathbf{vN}^{\mathrm{op}}$ is not
- Understand the interpretation of $\multimap$ better

Conclusions

Von Neumann algebras are powerful enough to interpret Selinger & Valiron's Quantum Lambda Calculus, via the adjunctions:

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}} \arrow[l, bend left, "\mathbf{vN}(-,\mathbb{C})"] \arrow[r, hook, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}} \arrow[l, bend left, "\mathcal{F}"]
\end{tikzcd}$$

Future work:

- Recursion
  - $\mathbf{vN}^{\mathrm{op}}_{\mathrm{CPsU}}$ is dcpo-enriched, but $\mathbf{vN}^{\mathrm{op}}$ is not
- Understand the interpretation of $\multimap$ better

Thank you!