Cohomological Framework for Contextual Quantum Computations

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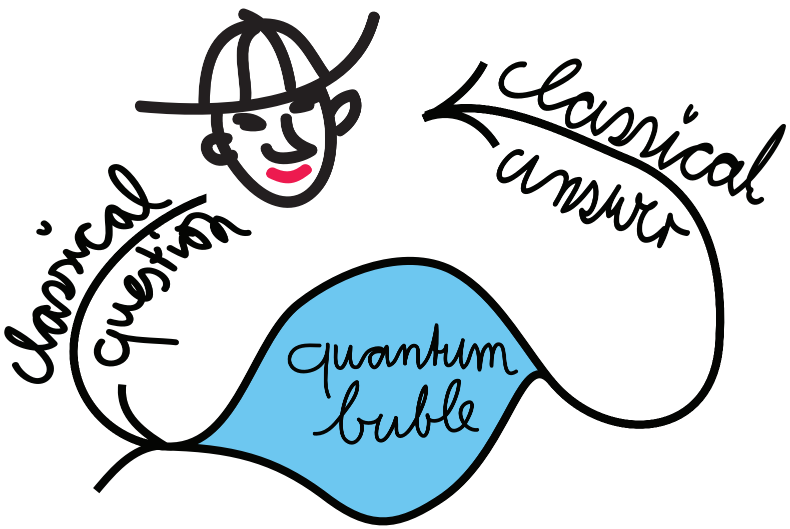
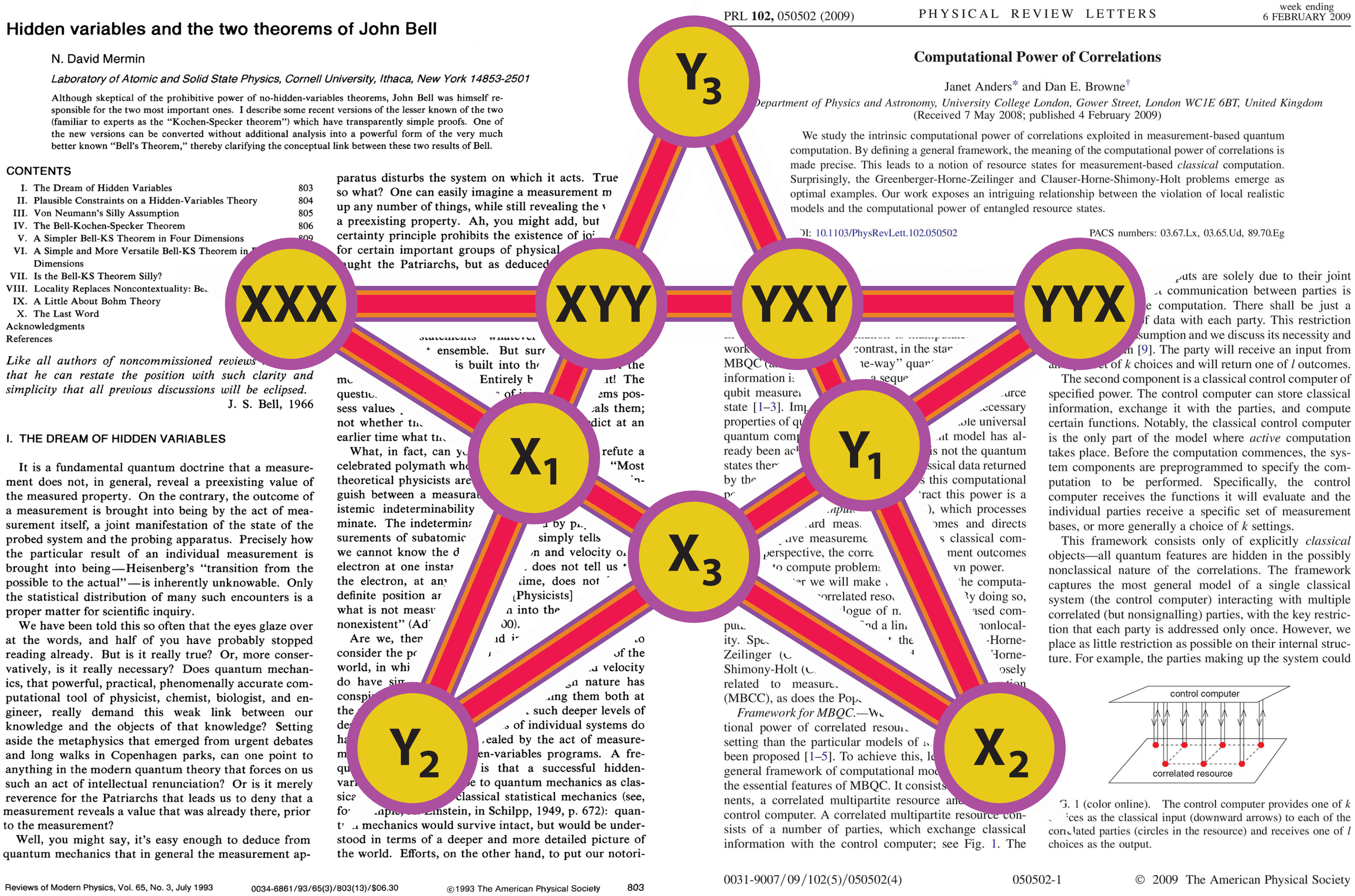
Computational structures in Hilbert space



Which fundamental computational structures exist in Hilbert space?

Two criteria:

- Must specify a classical input structure, a classical output structure, and a function computed.
- Must be genuinely quantum.



Hidden variables and the two theorems of John Bell

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Although skeptical of the prohibitive power of no-hidden-variables theorems, John Bell was himself responsible for the two most important ones. I describe some recent versions of the lesser known of the two (familiar to experts as the "Kochen-Specker theorem") which have transparently simple proofs. One of the new versions can be converted without additional analysis into a powerful form of the very much better known "Bell's Theorem," thereby clarifying the conceptual link between these two results of Bell.

CONTENTS

I. The Dream of Hidden Variables 803
II. Plausible Constraints on a Hidden-Variables Theory 804
III. Von Neumann's Silly Assumption 805
IV. The Bell-Kochen-Specker Theorem 806
V. A Simpler Bell-KS Theorem in Four Dimensions
VI. A Simple and More Versatile Bell-KS Theorem in
Dimensions
VII. Is the Bell-KS Theorem Silly?
VIII. Locality Replaces Noncontextuality: Be
IX. A Little About Bohm Theory
X. The Last Word
Acknowledgments
References

Like all authors of noncommissioned reviews ... that he can restate the position with such clarity and simplicity that all previous discussions will be eclipsed.
J. S. Bell, 1966

I. THE DREAM OF HIDDEN VARIABLES

It is a fundamental quantum doctrine that a measurement does not, in general, reveal a preexisting value of the measured property. On the contrary, the outcome of a measurement is brought into being by the act of measurement itself, a joint manifestation of the state of the probed system and the probing apparatus. Precisely how the particular result of an individual measurement is brought into being—Heisenberg's "transition from the possible to the actual"—is inherently unknowable. Only the statistical distribution of many such encounters is a proper matter for scientific inquiry.

We have been told this so often that the eyes glaze over at the words, and half of you have probably stopped reading already. But is it really true? Or, more conservatively, is it really necessary? Does quantum mechanics, that powerful, practical, phenomenally accurate computational tool of physicist, chemist, biologist, and engineer, really demand this weak link between our knowledge and the objects of that knowledge? Setting aside the metaphysics that emerged from urgent debates and long walks in Copenhagen parks, can one point to anything in the modern quantum theory that forces on us such an act of intellectual renunciation? Or is it merely reverence for the Patriarchs that leads us to deny that a measurement reveals a value that was already there, prior to the measurement?

Well, you might say, it's easy enough to deduce from quantum mechanics that in general the measurement apparatus disturbs the system on which it acts. True so what? One can easily imagine a measurement m up any number of things, while still revealing the v a preexisting property. Ah, you might add, but certainty principle prohibits the existence of joi for certain important groups of physical ught the Patriarchs, but as deduced

Computational Power of Correlations

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We study the intrinsic computational power of correlations exploited in measurement-based quantum computation. By defining a general framework, the meaning of the computational power of correlations is made precise. This leads to a notion of resource states for measurement-based *classical* computation. Surprisingly, the Greenberger-Horne-Zeilinger and Clauser-Horne-Shimony-Holt problems emerge as optimal examples. Our work exposes an intriguing relationship between the violation of local realistic models and the computational power of entangled resource states.

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... puts are solely due to their joint ... communication between parties is ... computation. There shall be just a ... of data with each party. This restriction ... sumption and we discuss its necessity and ... [9]. The party will receive an input from ... of k choices and will return one of l outcomes.

The second component is a classical control computer of specified power. The control computer can store classical information, exchange it with the parties, and compute certain functions. Notably, the classical control computer is the only part of the model where *active* computation takes place. Before the computation commences, the system components are preprogrammed to specify the computation to be performed. Specifically, the control computer receives the functions it will evaluate and the individual parties receive a specific set of measurement bases, or more generally a choice of k settings.

This framework consists only of explicitly *classical* objects—all quantum features are hidden in the possibly nonclassical nature of the correlations. The framework captures the most general model of a single classical system (the control computer) interacting with multiple correlated (but nonsignalling) parties, with the key restriction that each party is addressed only once. However, we place as little restriction as possible on their internal structure. For example, the parties making up the system could

Framework for MBQC.—W... tional power of correlated resou... setting than the particular models of ... been proposed [1–5]. To achieve this, ... general framework of computational mod... the essential features of MBQC. It consists ... nents, a correlated multipartite resource an... control computer. A correlated multipartite resource consists of a number of parties, which exchange classical information with the control computer; see Fig. 1. The

FIG. 1 (color online). The control computer provides one of k choices as the classical input (downward arrows) to each of the correlated parties (circles in the resource) and receives one of l choices as the output.

Contextuality, Cohomology & Computation

What happens if we combine those two links?

Results

- Introduce a cohomological framework for MBQC, based on the notion of a phase function.

  The phase function Φ has the following properties:

  $\rightarrow$ It is a 1-chain in group cohomology.

  $\rightarrow$ Contains the output function.

  $\rightarrow$ $d\Phi \neq 0$ is a witness for contextuality.

- For any G-MBQC, there is a non-contextuality inequality which bounds the cost of classical function evaluation.

- G-MBQCs classifiable by group cohomology: $H^2(G, N)$.

Outline

1. Review: Contextuality and measurement-based quantum computation (MBQC)

2. Cohomological formulation of MBQC

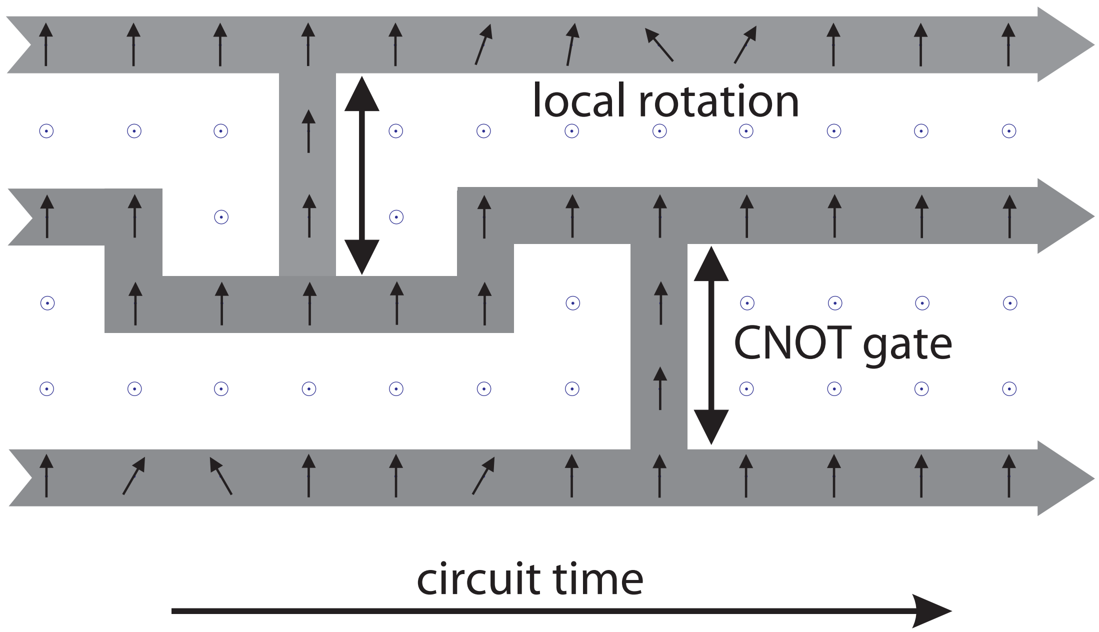
3. Ramifications of cohomology: contextuality/computation

4. Summary & open questions

Contextuality and MBQC

- Review: Contextuality and MBQC
- G-MBQC

Quantum computation by measurement



- Information written onto a cluster state, processed and read out by one-qubit measurements only.
- The resulting computational scheme is *universal*.

R. Raussendorf and H.-J. Briegel, PRL 86, 5188 (2001).

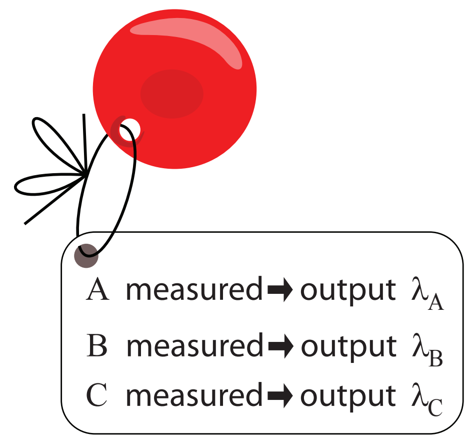
Contextuality of QM

What is a non-contextual hidden-variable model?

quantum mechanics

$$|\Psi\rangle$$

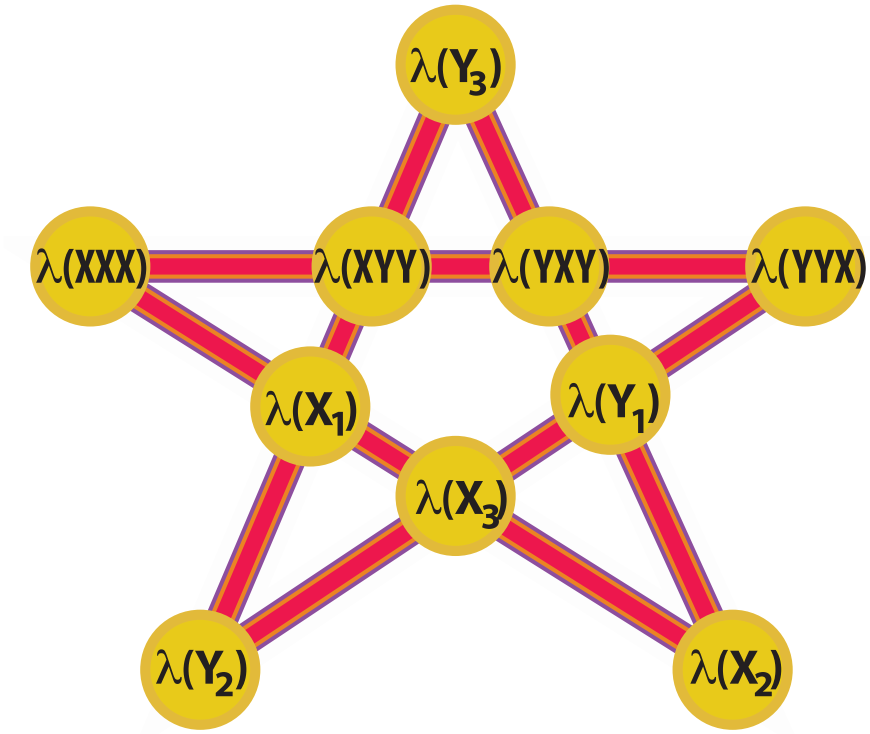
hidden-variable model



Noncontextuality: Given observables A,B,C: $[A, B] = [A, C] = 0$:
λ_A is *independent* of whether A is measured jointly with B or C.

Theorem [Kochen, Specker]: For $\dim(\mathcal{H}) \geq 3$, quantum-mechanics cannont be reproduced by a non-contextual hidden-variable model.

Simplest example: Mermin's star



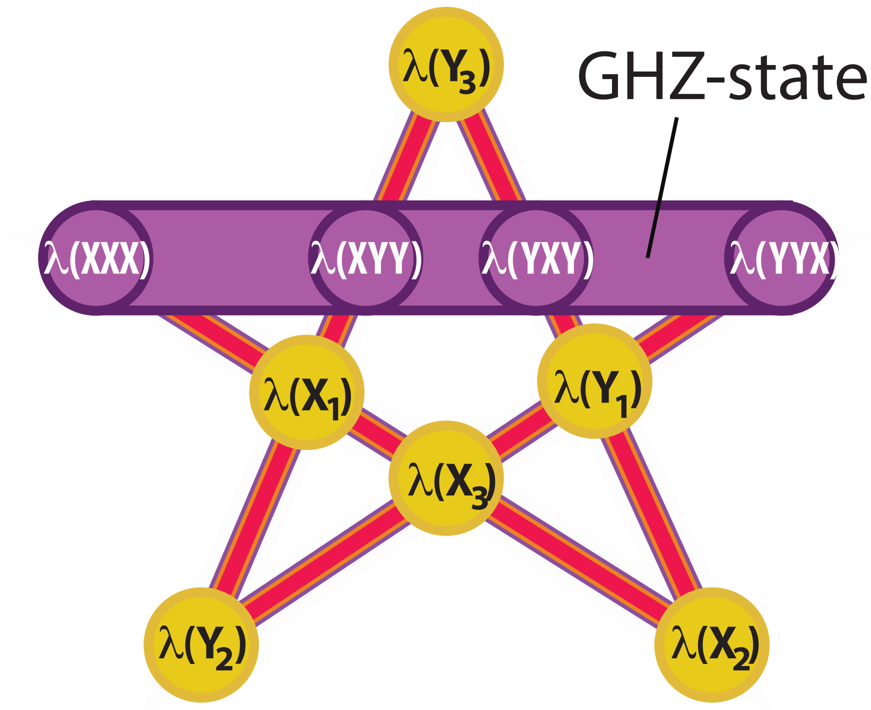
Is there a consistent value assignment $\lambda(\cdot) = \pm 1$ *for all observables in the star?*

- No consistent non-contextual value assignment λ exists.

  Any attempt to assign values leads to an algebraic contradiction.

N.D. Mermin, RMP 1992.

Simplest example: Mermin's star

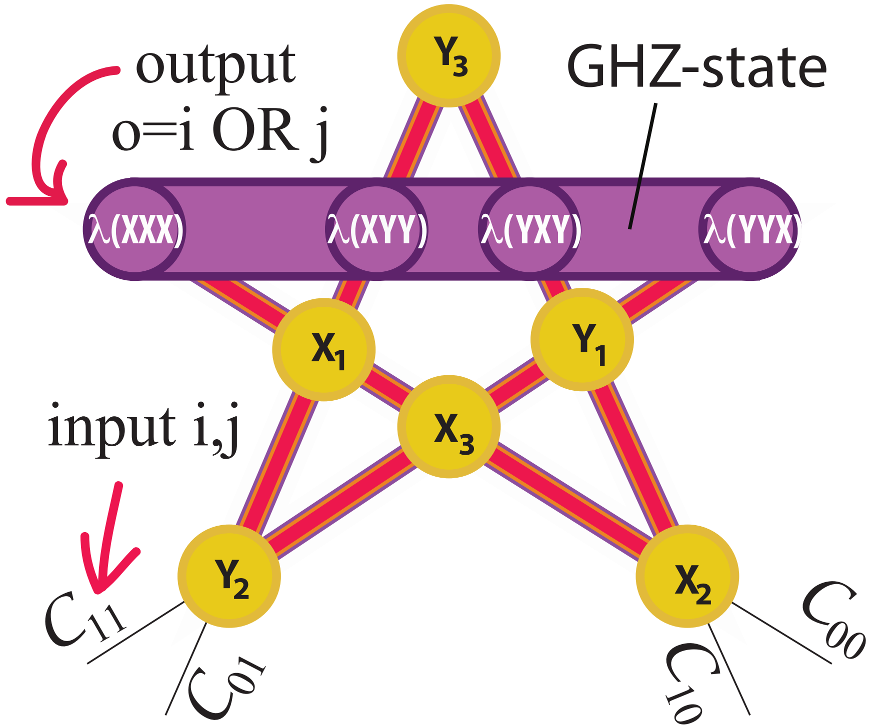


Mermin's star has a state-dependent version, invoking a GHZ-state.

- Still no consistent value assignment λ for the remaining local observables.

N.D. Mermin, RMP 1992.

Mermin's star computes

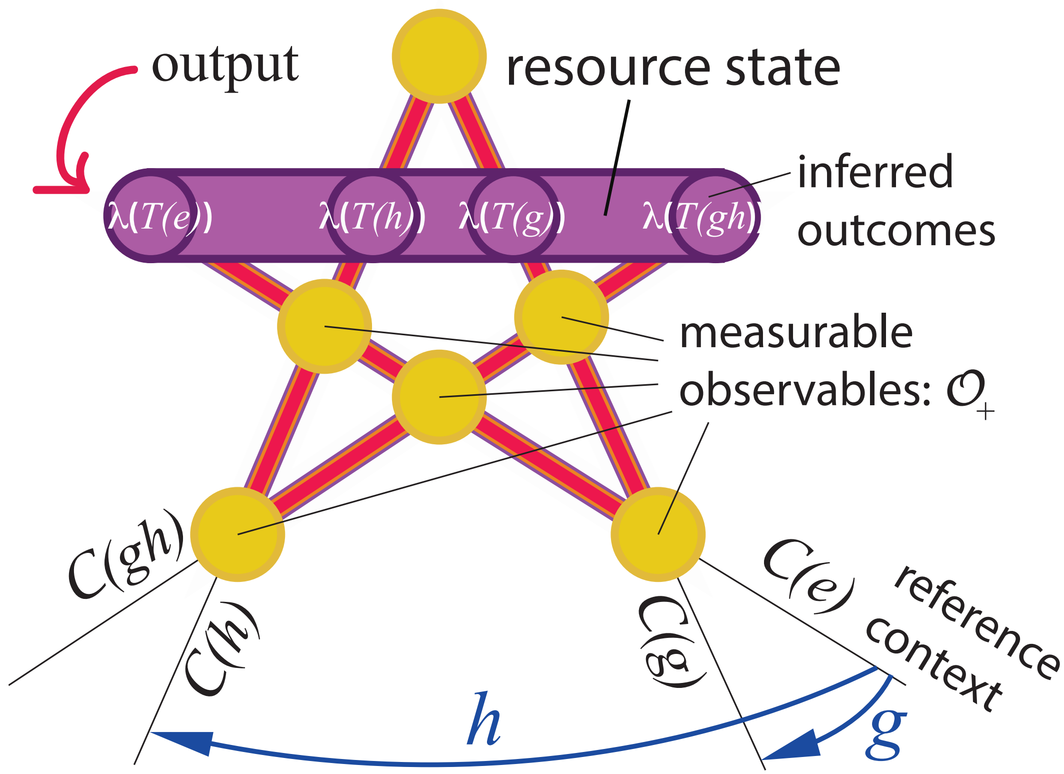


- Measurement contexts are assigned input values.
- Classical pre- and post-processing is mod 2 linear.
- Outputted OR-gate is *non-linear*.

- Extremely limited classical control computer is boosted to classical universality.

J. Anders and D. Browne, PRL 2009.

G-MBQC



- All observables $T \in \Omega_+$ have eigenvalues ± 1 only.
- Input values are elements of a group G.
- Outputted function is: $o: G \longrightarrow \mathbb{Z}_2$

Measurement context $C(g)$, given the input $g \in G$:

$$C(g) = \{u(g) T_a u(g)^\dagger,\ T_a \in C(e)\}.$$

Why this generalization?

- Some constraint on input set is required.

  Otherwise: Can put enormous computational power into the relation between input values and measurement contexts.

- G-MBQC contains standard MBQC as a special case.

Mind the specialization:

- Present analysis for temporally flat MBQCs only.

This setting we call G-MBQC

G-MBQC and the phase function

(a) The phase function Φ

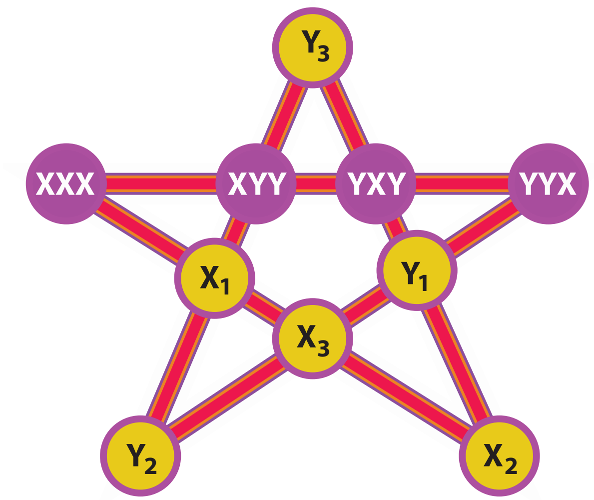
(b) Physical and computational ramifications

The phase function Φ

Recall the observables of interest:

- Measurable observables $T_a \in \mathcal{O}_+$
- Inferable observables $T(g)$, $g \in G$

$\rightarrow$ All of those: $\Omega_+ = \{T_a,\, a \in \mathcal{A}\}$.

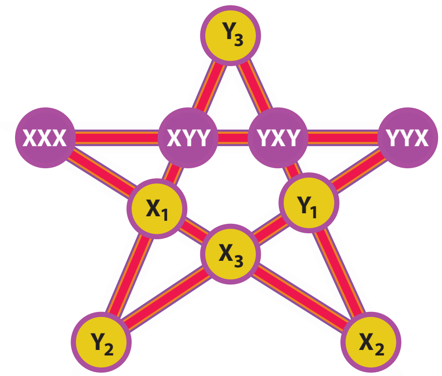


All admissible resource states ρ satisfy a symmetry condition:

$$\boxed{\langle T_{ga}\rangle_\rho = (-1)^{\Phi_g(a)}\langle T_a\rangle_\rho,\ \forall a \in \mathcal{A}.} \quad (1)$$

Therein, $T_{ga} := gT_ag^\dagger$, and Φ *is the phase function*.

The phase function Φ



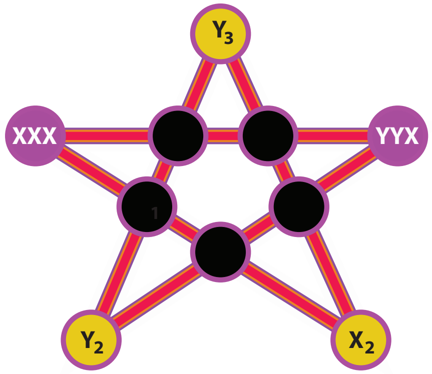
All admissible resource states ρ satisfy a symmetry condition:

$$\boxed{\langle T_{ga} \rangle_\rho = (-1)^{\Phi_g(a)} \langle T_a \rangle_\rho,\ \forall a \in \mathcal{A}.}$$

Therein, $T_{ga} := g T_a g^\dagger$, and Φ *is the phase function*.

- Check the GHZ case!
- The invariance condition Eq. (1) is satisfied for all G-MBQCs *on stabilizer states* which have uniform success probability.

The phase function Φ



The phase function Φ is a 1-chain in group cohomology,

$$\Phi : G \longrightarrow V.$$

V: module of consistent flips of observables

$$T_a \longrightarrow (-1)^{\mathbf{v}(a)} T_a,\ \mathbf{v} \in V$$

that preserve all product relations among commuting observables.

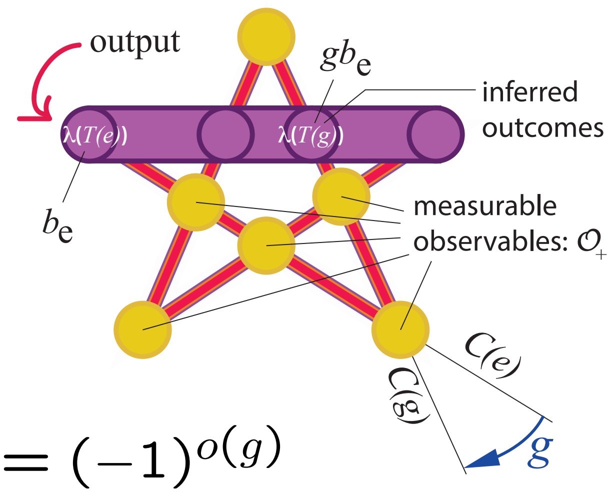
Linearity of Φ: For all T_a, T_b, T_c with $[T_a, T_b] = 0$ and $T_c = \pm T_a T_b$ it holds that

$$\Phi_g(c) = \Phi_g(a) + \Phi_g(b) \mod 2,\ \ \forall g \in G.$$

Ramifications of the cohomological framework

(a) Phase function and computation
(b) Cohomology and contextuality
(c) Contextuality and speedup

Phase function and computation



- Consider output observables $T_{g\,b_e} = T(g)$.
- Deterministic case (for simplicity): $\langle T_{gb_e}\rangle_\rho = (-1)^{o(g)}$

Recall the symmetry condition: $\langle T_{ga}\rangle_\rho = (-1)^{\Phi_g(a)}\langle T_a\rangle_\rho,\ \forall a \in \mathcal{A}$.

Hence, the output function $o: G \longrightarrow \mathbb{Z}_2$ is

$$\boxed{o(g) = \Phi_g(b_e) + o(e)}. \tag{2}$$

Phase function specifies output up to additive constant ① ✓

Cohomology and contextuality

Which phase functions are compatible with non-contextual hidden variable models (ncHVMs)?

Proposition 1. For any G-MBQC $\mathcal{M}$, if for all phase functions Φ satisfying the output relation $o(g) = \Phi_g(b_e) + c$ it holds that $d\Phi \not\equiv 0$, then $\mathcal{M}$ is contextual.

The group compatibility condition

Recall: $\boxed{\langle T_{ga}\rangle_\rho = (-1)^{\Phi_g(a)}\langle T_a\rangle_\rho,\ \forall a\in\mathcal{A}}$.

Multiplication is compatible: $\langle T_{gha}\rangle_\rho = \langle T_{(gh)a}\rangle_\rho = \langle T_{g(ha)}\rangle_\rho$

This implies:

$$(-1)^{\Phi_{gh}(a)}\langle T_a\rangle_\rho = (-1)^{\Phi_h(a)+\Phi_g(ha)}\langle T_a\rangle_\rho,$$

which can be satisfied in two ways. Either

$$\langle T_a\rangle_\rho = 0, \text{ or}$$

$$\boxed{\Phi_h(a)+\Phi_g(ha)-\Phi_{gh}(a) \mod 2 = 0}. \qquad (3)$$

Eq. (3) is the *group compatibility condition*. May be written as

$$(d\Phi)_{g,h}(a) = 0.$$

Cohomology and contextuality

Proposition 1. For any G-MBQC $\mathcal{M}$, if for all phase functions Φ satisfying the output relation $o(g) = \Phi_g(b_e) + c$ it holds that $d\Phi \not\equiv 0$, then $\mathcal{M}$ is contextual.

Proof: $\exists$ ncHVM $\Longrightarrow$ $\exists$ consistent value assignment s

Define a phase function $\Phi^{(s)}$ via $\Phi_g^{(s)}(a) := s(ga) - s(a) \mod 2$.

$\Phi^{(s)}$ satisfies the output relation $o(g) = \Phi_g(b_e) + c$, & $d\Phi \equiv 0$. $\Box$

The phase function contains a witness of quantumness

Symmetry-based contextuality proof for M's star

Recall: $X_1X_2X_3|\Psi\rangle = -X_1Y_2Y_3|\Psi\rangle = -Y_1X_2Y_3|\Psi\rangle = -Y_1Y_2X_3|\Psi\rangle = |\Psi\rangle$.

Consider: $G \ni g$ which transforms $X_1 \leftrightarrow Y_1$, $X_2 \leftrightarrow Y_3$, $X_3 \circlearrowleft$, $Y_3 \circlearrowleft$.

With the above eigenvalue equations we then have

$$\Phi_g(a_{XXX}) = 1,\ \Phi_g(a_{YXY}) = 0.$$

By linearity of Φ_g on commuting observables (definition of V),

$$\begin{array}{rcl}\Phi_g(a_{XXX}) &=& \Phi_g(a_{X_1}) + \Phi_g(a_{X_2}) + \Phi_g(a_{X_3}),\\ \Phi_g(a_{YXY}) &=& \Phi_g(a_{Y_1}) + \Phi_g(a_{X_2}) + \Phi_g(a_{Y_3}),\end{array}$$

where addition is mod 2. Adding those and using the former equation,

$$1 = \Phi_g(a_{X_1}) + \Phi_g(a_{X_3}) + \Phi_g(a_{Y_1}) + \Phi_g(a_{Y_3}).$$

The r.h.s. can be rewritten as a sum of coboundaries

$$1 = (d\Phi)_{g_{10},g_{01}}(a_{X_1}) + (d\Phi)_{g_{01},g_{10}}(a_{X_1}) + (d\Phi)_{g_{01},g_{01}}(a_{X_3}),$$

with $g_{01}, g_{10} \in G$.

Hence, $d\Phi \not\equiv 0$. With Prop 1., the state dependent Mermin star is contextual.

Contextuality and speedup

Proposition 2. The classical computational cost C_{class} of reducing the evaluation a function $o: G \to \mathbb{Z}_2$ to the evaluation of $o' : G \longrightarrow \mathbb{Z}_2$ compatible with an ncHVM is bounded by the maximum violation $\Delta(o)_{\text{max}}$ of a logical non-contextuality inequality

$$C_{\text{class}} \leq \Delta(o)_{\text{max}}.$$

Remark: The trivial function o' can be computed by the CC without any quantum resources, with memory of size $|\mathcal{O}_+|$.

Speedup requires significant room $\Delta(o)$ for violation of the logical contextuality inequality.

Summary

The following holds for all temporally flat G-MBQCs:

- The phase function is a 1-cochain in group cohomology.

  It describes what's being computed, and provides a witness for quantumness.

- For each G-MBQC exists a non-contextuality inequality that upper-bounds the hardness of classical function evaluation.

- G-MBQCs classifiable by group cohomology: $H^2(G, N)$.

arXiv:1602.04155

The next questions

- How do the above results extend to the temporally ordered case?

- Group cohomology has entered MBQC in a different vein, namely via "computational phases of matter". Is there a physical relation?

- Is there a quantum computational paradigm that relates to contextuality in the same way as "quantum parallelism" relates to superposition and interference?

arXiv:1602.04155

Additional material

Contextuality and speedup

The quantity

$$\boxed{\mathcal{W}(o)_\rho := \sum_{g\in G}(1+(-1)^{o(g)}\langle T(g)\rangle_\rho)/2}$$

is a contextuality witnesses.

- Maximum QM value: $\max(\mathcal{W}(o)) = |G|$.
- Maximum HVM value: $\max(\mathcal{W}(o)) = |G| - \Delta(o)$, with

$$\Delta(o) = \min_{s\in\mathcal{S}}\Big(\text{wt}(o \oplus o_s)\Big). \tag{4}$$